NONCROSSING PARTITIONS FROM HULL CONFIGURATIONS

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Abstract. Each finite configuration of points in the plane determines a corresponding lattice of noncrossing partitions. When these points form the vertex set of a convex polygon, the associated lattice is the classical noncrossing partition lattice (introduced by Kreweras in 1972), which makes many appearances in combinatorics and geometric group theory. If, on the other hand, all points of the configuration lie on a common line segment, the result is a Boolean lattice. In this article, we examine the more general class of *hull configurations*, which we define to be those which lie on the boundary of either a line segment or a convex polygon. We prove that the corresponding lattices of noncrossing partitions are unions of maximal Boolean subposets and, under certain circumstances, have symmetric chain decompositions.

Introduction

If P is the vertex set of a convex n-gon in the plane, then a set partition of P is *crossing* if there are two parts of the partition with overlapping convex hulls, or *noncrossing* otherwise. The *classical lattice of noncrossing partitions* $\mathrm{NC}(n)$ is the set of all noncrossing partitions of P, partially ordered by refinement. This poset was introduced by Kreweras in 1972 [Kre72] and has since become an important object in areas such as algebraic combinatorics, geometric group theory, and free probability. See [BBG$^+$19] and [McC06] for surveys.

By choosing a different starting configuration P (which is not necessarily the vertex set of a convex polygon) and considering the corresponding poset of noncrossing partitions $\mathrm{NC}(P)$, one obtains a natural generalization of $\mathrm{NC}(n)$. As in the classical case, $\mathrm{NC}(P)$ is a lattice for every choice of P [CDHM26], but other properties vary. For example, recent work of the authors with Fang, Jiang, Lin, Lindenmuth and Pokras [DFJ$^+$26] shows that when the configuration P lies on either a semicircle or a pair of rays with a common origin, then the resulting lattice $\mathrm{NC}(P)$ is graded and rank-symmetric; in particular, it has a symmetric chain decomposition.

In this article, we examine a larger class of configurations which includes the cases studied in [DFJ$^+$26]. In brief, we define a *hull configuration* to be one which lies either on a line segment or on the boundary of a convex n-gon. Intuitively, the noncrossing partition lattices for these configurations interpolate between the Boolean lattice (i.e. the lattice of noncrossing partitions for a configuration which lies on a line segment) and the classical lattice of noncrossing partitions. We also define a poset $\mathcal{H}(n)$ of all hull configurations with n points, up to an equivalence relation which allows for deformations of the configuration without disturbing collinearities. Minimal elements of $\mathcal{H}(n)$ are those which lie on a line segment, and maximal elements are those which lie on the vertex set of a convex n-gon. Each maximal chain

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in $\mathcal{H}(n)$ consists of n equivalence classes of configurations, which can be viewed as beginning with the vertex set of a polygon, then introducing collinearities one at a time until the resulting configuration lies on a line segment.

Our first main theorem extends the aforementioned result on symmetric chain decompositions to hull configurations P which have a "blank side," i.e. an edge of the convex hull which contains only two points of P. Boolean lattices were shown to have symmetric chain decompositions in a 1951 article by de Bruijn, van Ebbenhorst Tengbergen and Kruyswijk [dBvETK51], and the same was shown for the classical noncrossing partition lattice $\mathrm{NC}(n)$ by Simion and Ullman in 1999 [SU91]. Rephrased using our terminology, these articles showed that $\mathrm{NC}(P)$ admits a symmetric chain decomposition when P is either a minimal element of $\mathcal{H}(n)$ (in which case $\mathrm{NC}(P)$ is isomorphic to $\textsc{Bool}(n-1)$) or a maximal element of of $\mathcal{H}(n)$ (in which case $\mathrm{NC}(P)$ is isomorphic to $\mathrm{NC}(n)$). In [DFJ$^+$26], this result was extended to hull configurations referred to as cones and semicircles; the present article provides an even greater generalization.

Theorem A (Theorem 2.7)**.** *If P is a hull configuration with at least one blank side, then $\mathrm{NC}(P)$ has a symmetric chain decomposition.*

Based on several examples of hull configurations with no blank sides, it seems likely that Theorem A can be strengthened, as we record in the following conjecture.

Conjecture. *If P is a hull configuration with no blank sides, then $\mathrm{NC}(P)$ is not rank-symmetric (and therefore does not have a symmetric chain decomposition).*

Our second main theorem concerns the maximal Boolean subposets of $\mathrm{NC}(P)$. In the classical case, it was previously shown by T. Brady and McCammond that $\mathrm{NC}(n)$ is a union of maximal Boolean subposets [BM10]. In particular, it was shown by Haettel, Kielak and Schwer [HKS16] and McCammond [McC17] that the maximal Boolean subposets of $\mathrm{NC}(n)$ are in one-to-one correspondence with noncrossing trees with n vertices. More information on this correspondence and its relationship to the theory of buildings can be found in the work of Heller and Schwer [HS18], and in Heller's doctoral dissertation [Hel18].

In our more general setting, we say that a tree τ with vertex set P has *convex geodesics* if, for each side of $\textsc{Conv}(P)$, the unique geodesic in τ between the two endpoints of that side passes through every other point of P which lies in the interior of that side.

Theorem B (Theorem 3.6 and Theorem 3.9)**.** *If P is a hull configuration, then $\mathrm{NC}(P)$ is a union of maximal Boolean subposets, which are in one-to-one correspondence with the noncrossing trees on P with convex geodesics.*

In particular, this result has an interesting connection with the intrinsic geometry of the n-strand braid group. If we let $\overline{\mathrm{NC}(n)}$ denote the classical lattice of noncrossing partitions $\mathrm{NC}(n)$ with its maximum and minimum elements removed, then the order complex $|\overline{\mathrm{NC}(n)}|_\triangle$ (also known as the "diagonal link" of $\mathrm{NC}(n)$) is an $(n-3)$-dimensional ordered simplicial complex whose k-dimensional simplices correspond to chains in $\overline{\mathrm{NC}(n)}$ with $k+1$ elements. It was shown in [BM10] that, with the appropriate metric, $|\overline{\mathrm{NC}(n)}|_\triangle$ embeds in a spherical building in such a way that the image is a union of apartments. Moreover, Brady and McCammond conjectured that $|\overline{\mathrm{NC}(n)}|_\triangle$ is a CAT(1) metric space, which would imply that

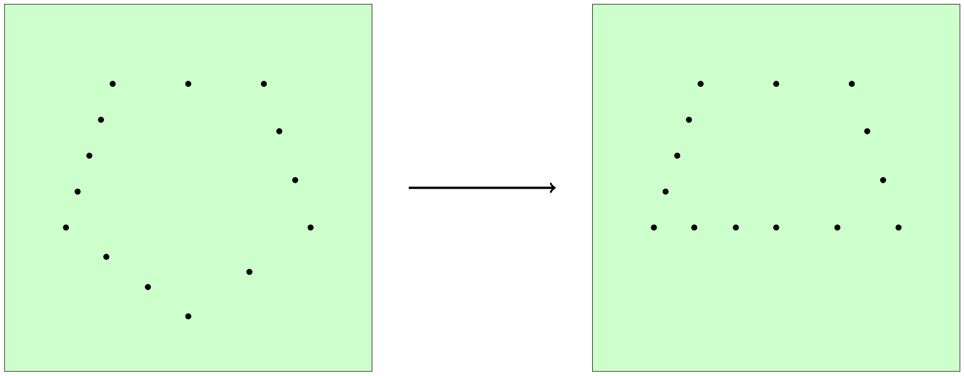

FIGURE 1. An elementary collapse m from a hull configuration Q (left) to another hull configuration $m(Q) = P$ (right)

the n-strand braid group is a CAT(0) group. This has been proven when $n \leq 7$ [BM10, HKS16, Jeo23], but remains open in general.

In this context, Theorem B tells us that when P is a hull configuration, the diagonal link of $\mathrm{NC}(P)$ is a union of apartments which can be viewed as a subcomplex of the diagonal link for $\mathrm{NC}(n)$. It would be interesting to know whether the smaller diagonal links found in $\mathrm{NC}(P)$ could be used to better understand the diagonal link for $\mathrm{NC}(n)$, and thus the intrinsic geometry of the braid group.

The article is organized into three sections. The first introduces our definitions for hull configurations and their associated partial order. The second section reviews symmetric chain decompositions and proves Theorem A; the third section includes the proof of Theorem B.

1. Convexity classes and hull configurations

In this section we introduce some basic terminology and properties for configurations. To start, each configuration of n points in $\mathbb{C}$ can be viewed as a single point in the *ordered configuration space* $\mathrm{Conf}_n(\mathbb{C})$, the topological subspace of $\mathbb{C}^n$ consisting of all n-tuples with distinct entries. Since we will primarily be interested in appearances of collinearities (rather than distances between points, for example), we introduce a useful equivalence relation on $\mathrm{Conf}_n(\mathbb{C})$.

Definition 1.1. For each configuration P in $\mathrm{Conf}_n(\mathbb{C})$, let $\mathcal{L}(P)$ be the arrangement of lines in $\mathbb{C}$ which pass through pairs of points in P and observe that $1 \leq |\mathcal{L}(P)| \leq \binom{n}{2}$. If a line in $\mathcal{L}(P)$ contains more than two points in P, then we say that the two extreme points on the line are *external collinearities* and the other points on the line are *internal collinearities*. For each $z \in P$, define the *multiplicity* $\ell(z)$ to be the number of lines for which z is an internal collinearity. Define the function $L\colon \mathrm{Conf}_n(\mathbb{C}) \to \mathbb{Z}^n$ by $L(z_1, \ldots, z_n) = (\ell(z_1), \ldots, \ell(z_n))$ and for each $P \in \mathrm{Conf}_n(\mathbb{C})$, let X_P denote the path component of $L^{-1}(L(P)) \subset \mathrm{Conf}_n(\mathbb{C})$ containing P. Finally, we define an equivalence relation on $\mathrm{Conf}_n(\mathbb{C})$ by declaring $P \sim Q$ if $X_P = X_Q$. We refer to the equivalence classes for this relation as *convexity classes*. For the remainder of the article, we will not distinguish between a configuration and the convexity class which contains it.

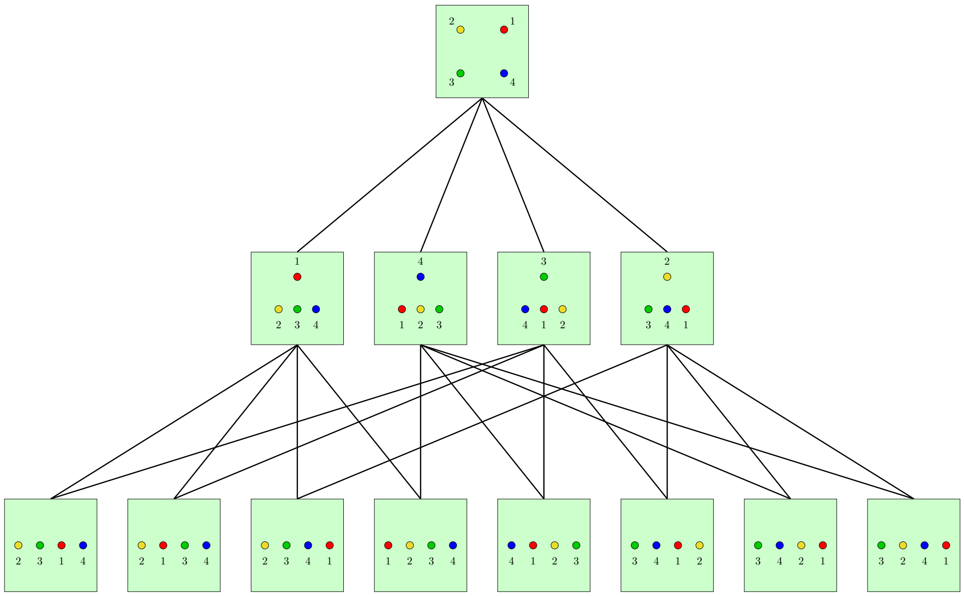

FIGURE 2. The lower set for a maximal element of $\mathcal{C}(4)$, the poset of convexity classes with four points.

The usual total order on the integers leads to a partial order on $\mathbb{Z}^n$ in the following manner: $(a_1, \ldots, a_n) \le (b_1, \ldots, b_n)$ if $a_i \le b_i$ for all i. Note that this is a graded lattice with rank function $\mathrm{rk}\colon \mathbb{Z}^n \to \mathbb{Z}$ given by $\mathrm{rk}(a_1, \ldots, a_n) = a_1 + \cdots + a_n$. This partial order can be pulled back through the map L to define a partial order on the set of convexity classes.

Definition 1.2. Let $\mathcal{C}(n)$ denote the set of all convexity classes with n points. Use the typical partial order on $\mathbb{Z}^n$ to define a partial order on $\mathcal{C}(n)$ as follows: declare $P \le Q$ in $\mathcal{C}(n)$ if $L(Q) \le L(P)$ in $\mathbb{Z}^n$ and the path component X_P is contained within the closure of the path component X_Q. In other words, $P \le Q$ if Q can be continuously deformed into P by (potentially) introducing additional collinearities without removing any. When $P < Q$ is a covering relation (i.e. there is no $R \in \mathcal{C}(n)$ with $P < R < Q$), we say that the deformation transforming Q into P is an *elementary collapse*; see Figure 1.

It is straightforward to see from the preceding definition that $\mathcal{C}(n)$ is a graded poset of height $n-2$ with rank function $\mathrm{rk}\colon \mathcal{C}(n) \to \mathbb{Z}$ given by defining

$$\mathrm{rk}(z_1, \ldots, z_n) = n - (\ell(z_1) + \cdots + \ell(z_n)).$$

Note that the minimal elements of $\mathcal{C}(n)$ have rank 2 and the maximal elements have rank n. In particular, every element of $\mathcal{C}(n)$ can be obtained from a maximal element via a finite sequence of elementary collapses. See Figures 2 and 3 for illustrations of parts of $\mathcal{C}(4)$.

Next, we introduce hull configurations, a special type of convexity class which form the main object of study in this article.

Definition 1.3. We say that $P \in \mathcal{C}(n)$ is a *hull configuration* if the convex hull $\textsc{Conv}(P)$ is either one-dimensional or contains no elements of P in its interior.

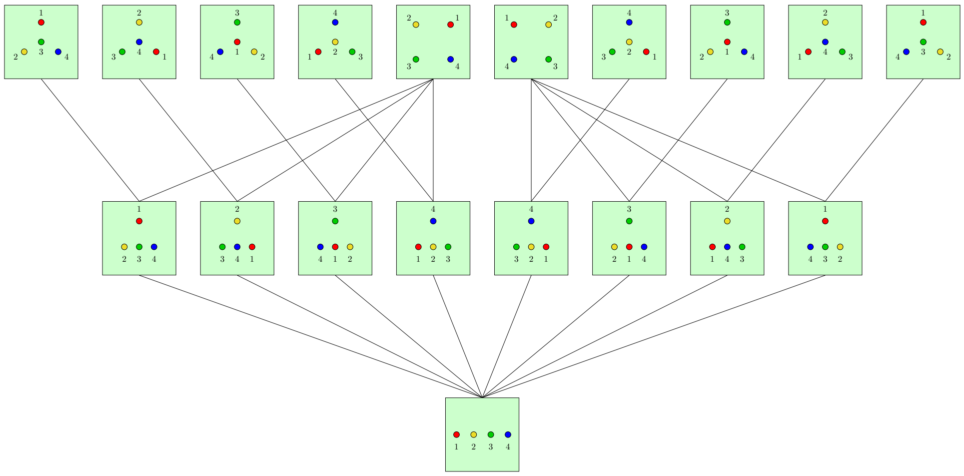

FIGURE 3. The upper set for a minimal element of $\mathcal{C}(4)$, the poset of convexity classes with four points. Note that only two of the ten maximal elements also belong to $\mathcal{H}(4)$, the poset of hull configurations with four points.

Observe that for each point z in a hull configuration P, the multiplicity $\ell(z)$ is either 0 or 1. Let $\mathcal{H}(n)$ denote the subposet of all hull configurations in $\mathcal{C}(n)$, and note that $\mathcal{H}(n)$ is a graded poset with the same rank function as $\mathcal{C}(n)$. In fact, there is an alternative definition in this setting: for all $P \in \mathcal{H}(n)$, $\mathrm{rk}(P) = 2$ if the convex hull $\textsc{Conv}(P)$ is one-dimensional, and otherwise $\mathrm{rk}(P) = k$, where k is the number of vertices in $\textsc{Conv}(P)$.

To record the number of internal collinearities in a hull configuration, we introduce the following notation.

Definition 1.4. Let P be a hull configuration of rank $k \geq 3$ in $\mathcal{H}(n)$. Number the sides of $\textsc{Conv}(P)$ from 1 to k counterclockwise, and let c_i denote the number of internal collinearities on the side numbered i. We define the *shape* of P to be $\textsc{Shape}(P) = [c_1; \ldots; c_k]$, the equivalence class of the k-tuple $(c_1, \ldots, c_k)$ up to cyclic shifts. In other words, $\textsc{Shape}(P)$ is a cyclic composition of $n-k$ into k parts (which may be zero). When $c_i = 0$, we say that side i is a *blank side* of the convex hull of P. Finally, we label the $c_i + 2$ points of P on side i as $z_{i,0}, z_{i,1}, \ldots, z_{i,c_i}, z_{i+1,0}$, where this linear order on side i comes from the counterclockwise cyclic order on P, and the first subscript is evaluated mod k. See Figure 4 for an illustration.

One can quickly see that $\mathcal{H}(n)$, like $\mathcal{C}(n)$, has $n!/2$ minimal elements, each of which corresponds to a unique way of linearly ordering n points up to reversal. In higher ranks, the number of elements is counted as follows.

Proposition 1.5. *Let $3 \leq k \leq n$. The number of elements of rank k in $\mathcal{H}(n)$ is $(n-1)!\binom{n}{k}$. In particular, $\mathcal{H}(n)$ has $(n-1)!$ maximal elements.*

Proof. A hull configuration of rank k is uniquely determined by two independent choices: one of the $(n-1)!$ different cyclic orders for the n points, and one of the $\binom{n}{k}$ different ways to choose k vertices from those n points. □

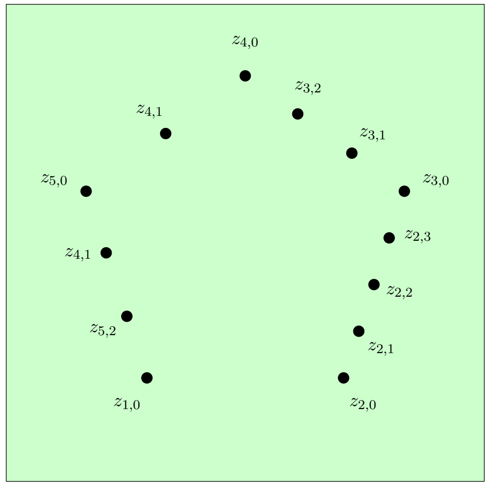

FIGURE 4. An illustration of our labeling conventions for a hull configuration with shape $[1; 2; 0; 3; 2]$.

This poset also satisfies some other interesting enumerative formulas, presented in the following propositions. Recall that for an element x in a poset R, the *lower set* is the subposet $\downarrow(x) = \{y \in R \mid y \leq x\}$ and the *upper set* is the subposet $\uparrow(x) = \{y \in R \mid x \leq y\}$.

Proposition 1.6. *Each minimal element of $\mathcal{H}(n)$ has 2^{n-2} maximal elements above it; each maximal element of $\mathcal{H}(n)$ has $n2^{n-3}$ minimal elements below it.*

Proof. For the first claim, observe that each minimum element of $\mathcal{H}(n)$ is a configuration P of n points, of which two are external and $n-2$ are internal collinearities. If we imagine these points arranged on a horizontal line segment, then each maximal configuration in $\mathcal{H}(n)$ above P is determined by a choice of moving each internal collinearity either above or below the line. Since these choices can be made independently of one another, there are 2^{n-2} maximal elements above P in $\mathcal{H}(n)$.

To see the second claim, recall that $\mathcal{H}(n)$ has $(n-1)!$ maximal elements and $n!/2$ minimal elements. By symmetry, this means that the number of minimal elements below a given maximum of $\mathcal{H}(n)$ is equal to $(n!/2)2^{n-2}/(n-1)!$, which simplifies to $n2^{n-3}$. □

Proposition 1.7. *If $P \leq Q$ in $\mathcal{H}(n)$, then the interval $[P, Q]$ is isomorphic to a Boolean lattice of height $rk(Q) - rk(P)$.*

Proof. Suppose that $P \leq Q$ and $rk(Q) - rk(P) = k$. Then there is a motion which takes Q to P while preserving all existing internal collinearities in Q and adding k new ones. These new collinearities can be added independently in any order, and so each element of $[P, Q]$ in $\mathcal{H}(n)$ is completely determined by a subset of these k internal collinearities. Thus the interval $[P, Q]$ is isomorphic to BOOL(k), the poset of subsets of a k-element set. □

2. SYMMETRIC CHAIN DECOMPOSITIONS AND RANK-SYMMETRY

In this section we introduce the poset of noncrossing partitions associated to a convexity class and examine some its properties. In particular, we show that the lattice $\mathrm{NC}(P)$ has a symmetric chain decomposition when the configuration P has a blank side, proving Theorem A. First, we give a definition for $\mathrm{NC}(P)$.

Definition 2.1. For each $P \in \mathrm{Conf}_n(\mathbb{C})$, a set partition of P into subsets is *noncrossing* if the convex hulls of the subsets are pairwise disjoint. The set of all partitions for P, denoted $\Pi(P)$, forms a lattice under the partial order of refinement, and the subset of all noncrossing partitions forms a subposet denoted $\mathrm{NC}(P)$. Note that there is a natural isomorphism between $\mathrm{NC}(P)$ and $\mathrm{NC}(Q)$ whenever P and Q belong to the same convexity class, so it is safe to define the lattice $\mathrm{NC}(P)$ for each $P \in \mathcal{C}(n)$.

As described in the introduction, $\mathrm{NC}(P)$ is isomorphic to the classical lattice of noncrossing partitions $\mathrm{NC}(n)$ when P is a maximal element of $\mathcal{H}(n)$. Next, we note that the partial order on $\mathcal{C}(n)$ corresponds to inclusion of the corresponding lattices of noncrossing partitions.

Proposition 2.2. *If $P \leq Q$ in $\mathcal{C}(n)$, then $\mathrm{NC}(P)$ is isomorphic to a subposet of $\mathrm{NC}(Q)$. In particular, if $P \in \mathcal{H}(n)$, then $\mathrm{NC}(P)$ is isomorphic to a subposet of the classical noncrossing partition lattice $\mathrm{NC}(n)$.*

Proof. If $P \leq Q$ in $\mathcal{C}(n)$, then P can be obtained from Q by a continuous motion $m\colon Q \to P$ which preserves all existing collinearities and potentially adds some new ones. This induces an isomorphism of partition lattices $m\colon \Pi(Q) \to \Pi(P)$, and no crossing partition of Q can become noncrossing by adding collinearities, which means that restricting the domain of m to $\mathrm{NC}(Q)$ yields an injective order-preserving map $m\colon \mathrm{NC}(Q) \to \mathrm{NC}(P)$. Thus, $\mathrm{NC}(Q)$ is isomorphic to a subposet of $\mathrm{NC}(P)$.

Moreover, if M is a maximal element of $\mathcal{H}(n)$, then it is represented by a configuration of n points in convex position, and thus $\mathrm{NC}(M)$ is isomorphic to the classical noncrossing partition lattice $\mathrm{NC}(n)$. It then follows that for all $P \in \mathcal{H}(n)$, we have that $\mathrm{NC}(P)$ is isomorphic to a subposet of $\mathrm{NC}(n)$. □

The poset $\mathrm{NC}(n)$ has a wealth of interesting properties; to name a few, it is *bounded* (it has a unique minimum $\hat{0}$ and a unique maximum $\hat{1}$), it is a *lattice* (for all $x, y \in \mathrm{NC}(n)$, there is a unique meet $x \wedge y$ and a unique join $x \vee y$), and it is *graded* with rank function $\mathrm{rk}\colon \mathrm{NC}(P) \to \mathbb{Z}$ given by defining $\mathrm{rk}(\pi)$ to be $n-k$, where k is the number of blocks in π.

Some of the properties satisfied by $\mathrm{NC}(n)$ are also satisfied by the more general definition of $\mathrm{NC}(P)$. For example, $\mathrm{NC}(P)$ is a bounded lattice for every choice of convexity class P [CDHM26, Proposition 2.3]. In addition, it was shown in [DFJ+26, Theorem 2.3] that when P is a hull configuration, $\mathrm{NC}(P)$ is graded. In the remainder of this section, we show that in certain circumstances, $\mathrm{NC}(P)$ has a symmetric chain decomposition, a property which is also shared by $\mathrm{NC}(n)$.

Definition 2.3. Let $P \in \mathcal{H}(n)$. A subset of $\mathrm{NC}(P)$ is *centered* if for each k, it has an equal number of elements at rank k and at rank $n-k$. A totally ordered subset of a poset is called a *chain*, and a chain $\pi_1 < \cdots < \pi_k$ is *saturated* if each $\pi_i < \pi_{i+1}$ is a covering relation. Finally, $\mathrm{NC}(P)$ has a *symmetric chain decomposition* if it can be expressed as the disjoint union of centered saturated chains.

Our approach for Theorem A is similar to those of Theorem 3.2 and Theorem 4.2 in [DFJ+26]. In particular, we use an inductive argument to express $\mathrm{NC}(P)$ as

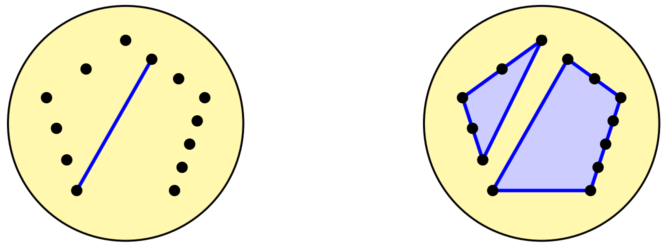

FIGURE 5. Elements $\alpha_{3,2}$ (left) and $\beta_{3,2}$ (right) for the configuration P depicted in Figure 4, as described in Definition 2.4.

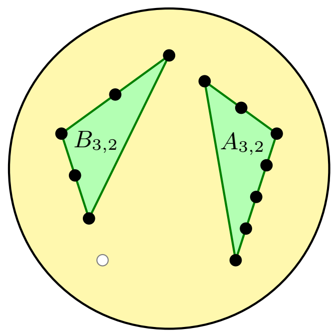

FIGURE 6. Subsets $A_{3,2}$ and $B_{3,2}$ of the configuration P depicted in Figure 4, as described in Definition 2.4. Note in particular that $A_{3,2} \cup B_{3,2} = P - \{z_{1,0}\}$.

the disjoint union of centered subposets, each of which has a symmetric chain decomposition. To this end, we introduce some notation for special elements and intervals in $\mathrm{NC}(P)$ when P has a blank side.

Definition 2.4. Suppose $P \in \mathcal{H}(n)$ has at least one blank side; without loss of generality, let $\textsc{Shape}(P) = [0; c_2; \cdots; c_k]$. For each choice of i and j such that $2 \leq i < k$ and $j \leq c_i$, we define two elements α_{ij} and β_{ij} of $\mathrm{NC}(P)$, as well as two subsets A_{ij} and B_{ij} of P. First, let α_{ij} be the unique atom in $\mathrm{NC}(P)$ with non-singleton block $\{z_{1,0}, z_{i,j}\}$, and let β_{ij} be the unique coatom in $\mathrm{NC}(P)$ such that one block contains all points from $z_{1,0}$ to $z_{i,j}$ in the counterclockwise order around the boundary of $\textsc{Conv}(P)$, and another (nonempty) block contains all other elements of P. See Figure 5 for an illustration and note that $\alpha_{ij} \leq \beta_{ij}$. Also, define A_{ij} to be the subset of P consisting of all points in the counterclockwise order from $z_{2,0}$ to $z_{i,j}$, and define $B_{ij} = P - (A_{ij} \cup \{z_{1,0}\})$; see Figure 6.

Lemma 2.5. *Suppose $P \in \mathcal{H}(n)$ has at least one blank side, and define the elements α_{ij} and β_{ij} and the subsets A_{ij} and B_{ij} as in Definition 2.4. Then the interval $[\alpha_{ij}, \beta_{ij}]$ is isomorphic to the product $\mathrm{NC}(A_{ij}) \times \mathrm{NC}(B_{ij})$.*

Proof. It is straightforward to see that $[\alpha_{ij}, \beta_{ij}]$ is isomorphic to the product of $\mathrm{NC}(B_{ij})$ and the subposet of $\mathrm{NC}(A_{ij} \cup \{z_{1,0}\})$ consisting of all partitions where $z_{1,0}$ and $z_{i,j}$ share a block. We know from [DFJ$^+$26, Lemma 2.6] that this subposet of $\mathrm{NC}(A_{ij} \cup \{z_{1,0}\})$ is isomorphic to $\mathrm{NC}(A_{ij})$, so it follows that $[\alpha_{ij}, \beta_{ij}]$ is isomorphic to $\mathrm{NC}(A_{ij}) \times \mathrm{NC}(B_{ij})$. ∎

Lemma 2.6. *Suppose* $P \in \mathcal{H}(n)$ *with* $\text{SHAPE}(P) = [0; c_2; \cdots; c_k]$. *Let* X *be the subposet of* $\text{NC}(P)$ *consisting of all elements in which* $z_{1,0}$ *is either a singleton or in a block containing* z_{k,c_k}. *Then* $X \cong \text{NC}(P - \{z_{1,0}\}) \times \text{BOOL}(1)$, *and* $\text{NC}(P)$ *is the disjoint union of* X *and the intervals of the form* $[\alpha_{ij}, \beta_{ij}]$ *where* $2 \le i \le k-1$ *and* $0 \le j \le c_i$.

Proof. For the first claim, note that the subposet of $\text{NC}(P)$ in which $z_{1,0}$ is a singleton is isomorphic to $\text{NC}(P - \{z_{1,0}\})$, as is the subposet of $\text{NC}(P)$ in which $z_{1,0}$ shares a block with z_{k,c_k}. Each element in the former subposet is covered by a unique element of the latter, obtained by combining the block containing z_{k,c_k} with the singleton block $\{z_{1,0}\}$. From here, it is straightforward to see that $X = \text{NC}(P - \{z_{1,0}\}) \times \text{BOOL}(1)$.

Next, let $\pi \in \text{NC}(P)$ and note that there are three distinct possibilities for $z_{1,0}$ in π. If $\{z_{1,0}\}$ is a singleton block of π, then $\pi \in X$. If $z_{1,0}$ shares a block with z_{k,c_k}, then we again have that $\pi \in X$. Otherwise, let $z_{i,j}$ be the last point in P which shares a block with $z_{1,0}$ with respect to the counterclockwise order on P which begins at $z_{1,0}$. Necessarily we have $2 \le i \le k-1$ and $0 \le j \le c_i$, and in particular π belongs to $[\alpha_{ij}, \beta_{ij}]$ and no other interval of this type. Therefore, $\text{NC}(P)$ can be expressed as the desired disjoint union. □

Theorem 2.7 (Theorem A)**.** *If* $P \in \mathcal{H}(n)$ *has at least one blank side, then* $\text{NC}(P)$ *has a symmetric chain decomposition.*

Proof. Without loss of generality, suppose that $\text{SHAPE}(P) = [0, c_2, \ldots, c_k]$. We will proceed by induction on n, the number of points in P. First, consider the base case $n = 3$. Then the elements of P lie either on a common line segment, in which case $\text{NC}(P) \cong \text{BOOL}(2)$, or on the vertices of a triangle, in which case $\text{NC}(P) \cong \text{NC}(3)$. In both cases, $\text{NC}(P)$ has a symmetric chain decomposition by inspection.

Now suppose that the theorem holds for all configurations with at most $n-1$ points and at least one blank side. By Lemmas 2.5 and 2.6, we know that $\text{NC}(P)$ is the disjoint union of X, which is isomorphic to $\text{NC}(P - \{z_{1,0}\}) \times \text{BOOL}(1)$, and the intervals of the form $[\alpha_{ij}, \beta_{ij}]$, each of which is isomorphic to $\text{NC}(A_{ij}) \times \text{NC}(B_{ij})$. Since $P - \{z_{1,0}\}$, A_{ij}, and B_{ij} are configurations with at least one blank side and strictly fewer points than P, we know that their associated lattices of noncrossing partitions admit symmetric chain decompositions by the inductive hypothesis. Since the property of having a symmetric chain decomposition is preserved under direct products of posets, we can therefore say that $\text{NC}(P)$ is the disjoint union of centered subposets, each of which admits a symmetric chain decomposition, and it follows that $\text{NC}(P)$ has one as well. By induction, the claim holds for all hull configurations P with at least one blank side. □

3. Maximal Boolean subposets

In this section, we show that a correspondence between $\text{NC}(n)$ and noncrossing trees can be generalized to $\text{NC}(P)$, and then leverage this to prove Theorem B.

Definition 3.1. Let $P \in \mathcal{C}(n)$. A *noncrossing forest* on P is an embedded graph with no cycles in $\mathbb{C}$ with vertex set P and geodesic edges. If μ_1 and μ_2 are noncrossing forests with the same vertex set such that μ_1 is a subgraph of μ_2, then we write $\mu_1 \subseteq \mu_2$. A *noncrossing tree* is a connected noncrossing forest. Also, there is a natural bijection between the noncrossing forests for configurations in the same convexity class. Finally, if μ is a noncrossing forest on the vertex set P, define $\textsc{Part}(\mu)$ to be the partition of P in which distinct elements x and y share a block whenever they belong to the same connected component of μ.

If M is a maximal element of $\mathcal{H}(n)$, then $\mathrm{NC}(M) \cong \mathrm{NC}(n)$ and the noncrossing trees on M are enumerated by the Fuss–Catalan number $C^{(3)}_{n-1} = \frac{1}{2n-1}\binom{3n-3}{n-1}$. Moreover, it was previously shown that there is a bijection between these trees and the subposets of $\mathrm{NC}(n)$ which are isomorphic to $\textsc{Bool}(n-1)$, the Boolean lattice of height $n-1$.

Theorem 3.2 ([HKS16, Proposition 4.4] and [McC17, Proposition 5.6])**.** *Let M be a maximal element of $\mathcal{H}(n)$. Then there is a one-to-one correspondence between the maximal Boolean subposets of $\mathrm{NC}(M)$ and the noncrossing trees on M.*

The correspondence above can be extended to the more general case of $\mathrm{NC}(P)$, but we first need to define a special type of noncrossing tree.

Definition 3.3. Let $P \in \mathcal{H}(n)$. We say that a noncrossing forest μ on P has *convex geodesics* if for each subtree $\tau \subseteq \mu$, the only elements of P in $\textsc{Conv}(\tau)$ are the vertices of τ. Equivalently, for each pair of distinct vertices $x, y \in P$ which belong to the same connected component of μ, the unique path from x to y in μ has a convex hull which includes only the elements of P which lie along the path.

While we will not need it in this article, it can also be shown that a noncrossing tree τ on P has convex geodesics if and only all leaves of τ (i.e. the vertices of degree 1) lie on corners of $\textsc{Conv}(P)$.

Lemma 3.4. *If μ is a noncrossing forest on the vertex set P with convex geodesics, then $\textsc{Part}(\mu)$ is a noncrossing partition of P.*

Proof. Each pair of distinct blocks A_1 and A_2 in $\textsc{Part}(\mu)$ can be viewed as the vertex sets for distinct subtrees τ_1 and τ_2 in μ. Since μ has convex geodesics, we know that $\textsc{Conv}(A_i) \cap P = \textsc{Conv}(\tau_i) \cap P = A_i$, so the convex hull of each A_i does not contain any additional points of P. Moreover, the convex hulls of A_1 and A_2 do not have intersecting interiors since τ_1 and τ_2 do not overlap because μ is noncrossing. Therefore, $\textsc{Conv}(A_1)$ and $\textsc{Conv}(A_2)$ are disjoint, so $\textsc{Part}(\mu)$ is noncrossing. □

Lemma 3.5. *Let $\pi_1, \ldots, \pi_k$ be atoms in $\mathrm{NC}(P)$ such that the corresponding edges $e_1, \ldots, e_k$ form a tree with vertex set $Q \subseteq P$. Then $\mathrm{rk}(\pi_1 \vee \cdots \vee \pi_k) = k$ if and only if $\textsc{Conv}(e_1, \ldots, e_k) \cap P = Q$.*

Proof. Let $\pi \in \mathrm{NC}(P)$ denote the join $\pi_1 \vee \cdots \vee \pi_k$ and note that π is the partition of P which has $\textsc{Conv}(e_1, \ldots, e_k) \cap P$ as its unique non-singleton block. Since $Q \subseteq \textsc{Conv}(e_1, \ldots, e_k) \cap P$ and $|Q| = k + 1$ (because $e_1, \ldots, e_k$ form a tree), we see that $\mathrm{rk}(\pi) \leq k$, with equality precisely when $\textsc{Conv}(e_1, \ldots, e_k) \cap P = Q$. □

We now state and prove a generalization of Theorem 3.2, which forms the first part of Theorem B.

Theorem 3.6. *Let $P \in \mathcal{H}(n)$. Then there is a one-to-one correspondence between the maximal Boolean subposets of $\mathrm{NC}(P)$ and the noncrossing trees on P with convex geodesics.*

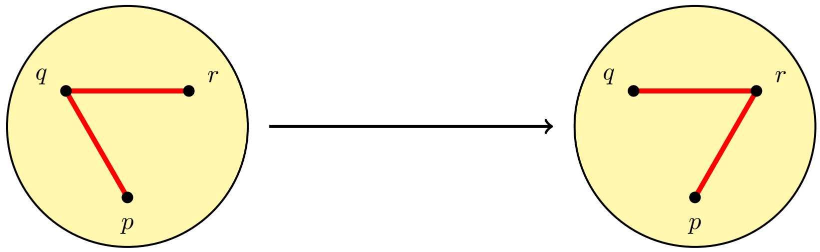

FIGURE 7. On the left, a tree on the vertex set $\{p, q, r\}$; on the right, the result of sliding the edge $\{p, q\}$ along the edge $\{q, r\}$.

Proof. First, let τ be a noncrossing tree on P with convex geodesics and recall that each subforest $\mu \subseteq \tau$ determines a noncrossing partition $\text{PART}(\mu)$ of P by Lemma 3.4. It is straightforward to see that the 2^{n-1} subforests of τ produce distinct partitions, and $\text{PART}(\mu_1) \subseteq \text{PART}(\mu_2)$ in $\text{NC}(P)$ if and only if $\mu_1 \subseteq \mu_2$, so it follows that each noncrossing tree τ with convex geodesics on P determines a maximal Boolean subposet (i.e. a copy of $\text{BOOL}(n-1)$) in $\text{NC}(P)$—call this subposet $\text{BOOL}(\tau)$. Since the atoms of $\text{BOOL}(\tau)$ correspond to the edges of τ, we can then see that the map $\tau \mapsto \text{BOOL}(\tau)$ is an injection from the set of noncrossing trees on P with convex geodesics to the set of maximal Boolean subposets of $\text{NC}(P)$.

Now, let $\mathcal{B}$ be a maximal Boolean subposet of $\text{NC}(P)$, i.e. $\mathcal{B} \cong \text{BOOL}(n-1)$. Then the maximum element of $\text{NC}(P)$, which consists of the single block P, is the join of the $n-1$ atoms of $\mathcal{B}$, and it follows that the edges corresponding to these atoms form a connected graph with vertex set P. Since there are $n-1$ edges and $|P| = n$, we know by the Euler characteristic that this connected graph is a tree, and since the join of any two atoms has rank 2 in $\text{BOOL}(n-1)$, this tree must be noncrossing—call it τ. In a Boolean lattice, the join of any k atoms has rank k, so by Lemma 3.5, every subtree of τ has a convex hull which intersects P in its vertex set. In other words, τ has convex geodesics. It follows that $\mathcal{B} = \text{BOOL}(\tau)$ and the proof is complete. $\square$

In addition to Theorem 3.2, it was shown in [HKS16] and [McC17] that $\text{NC}(n)$ is the union of its maximal Boolean subposets. To close this section, we prove the analogous result for $\text{NC}(P)$ when P is a hull configuration. First, we will need a way of perturbing noncrossing trees by "sliding" edges.

Definition 3.7. Let $P \in \mathcal{H}(n)$ and let τ be a noncrossing tree on P with convex geodesics. Suppose that p, q and r are vertices of τ and $e = \{p, q\}$, $f = \{q, r\}$ are edges of τ which are adjacent in the cyclic order of edges incident to q. If we replace e with a new edge $e' = \{p, r\}$, then the resulting graph τ' is still a tree on P, which we say was obtained by *sliding* e *along* f. See Figure 7 for an illustration.

Lemma 3.8. *Let $P \in \mathcal{H}(n)$, let τ be a noncrossing tree on P with convex geodesics, and suppose that τ' is another noncrossing tree on P with convex geodesics which is the result of sliding the edge $e = \{p, q\}$ along $f = \{q, r\}$. If $\pi \in \text{BOOL}(\tau)$, then π is also an element of $\text{BOOL}(\tau')$ if and only if the block containing p also contains either both q and r or neither q nor r.*

Proof. Suppose $\pi \in \text{BOOL}(\tau)$ and let A be the set of atoms in $\text{BOOL}(\tau)$ below π, i.e. the join of all elements in A is $\bigvee A = \pi$. Also, let π_{pq} and π_{pr} be the atoms of $\text{NC}(P)$

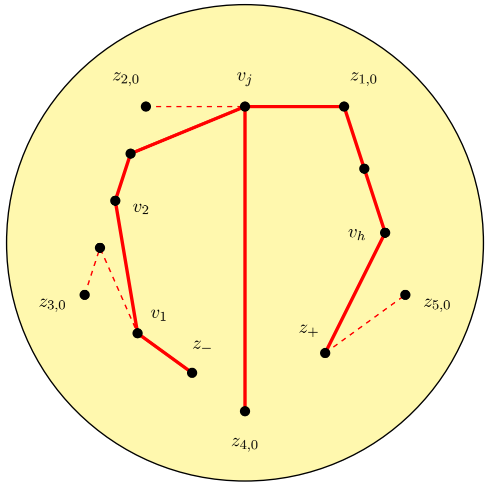

FIGURE 8. A noncrossing tree τ on a hull configuration Q, with the geodesic between vertices z_- and z_+ highlighted.

with unique non-singleton block $\{p, q\}$ and $\{p, r\}$, respectively. If $\pi_{pq} \notin A$, then π remains in $\text{BOOL}(\tau')$ since all elements of A remain in the new poset. Note that this corresponds to the case where p belongs to a distinct block from the block (or blocks) containing q and r, since $\pi_{pq} \not\leq \pi$ and $\pi_{pr} \not\leq \pi$. If on the other hand $\pi_{pq} \in A$ and $\pi \in \text{BOOL}(\tau')$, then it must be that $\pi_{pr} \leq \pi$, else π could be represented as the join of elements in $A - \{\pi_{pq}\}$, which contradicts our assumption. Conversely, if $\pi_{pq} \leq \pi$ and $\pi_{pr} \leq \pi$, then the join of all elements in $(A - \{\pi_{pq}\}) \cup \{\pi_{pr}\}$ must be equal to π, meaning that $\pi \in \text{BOOL}(\tau')$. Since this corresponds to the case where p, q and r share a block in π, the proof is complete. ☐

We are now ready to complete the proof of Theorem B.

Theorem 3.9. *For each $P \in \mathcal{H}(n)$, the lattice $\text{NC}(P)$ is a union of maximal Boolean subposets.*

Proof. From [HKS16] and [McC17], we already know that $\text{NC}(P)$ is a union of maximal Boolean subposets when P is a maximal element of $\mathcal{H}(n)$. Now, suppose that there is a hull configuration $Q \in \mathcal{H}(n)$ such that $\text{NC}(Q)$ is a union of maximal Boolean subposets, and let $P = m(Q)$ be the configuration obtained by applying the elementary collapse m to Q. Since every element of $\mathcal{H}(n)$ can be obtained by applying a finite sequence of elementary collapses to a maximal element, it suffices to show that $\text{NC}(P)$ is a union of maximal Boolean subposets.

Let $\pi \in \text{NC}(P)$ with $\text{SHAPE}(\pi) = [c_1; \cdots; c_k]$ and suppose that $z_{i,0}$ is the external vertex of Q which is made into an internal collinearity of P under the elementary collapse m. Then $\pi = m(\rho)$ for some $\rho \in \text{NC}(Q)$, and by combining Theorem 3.6 with the assumption that $\text{NC}(Q)$ is a union of maximal Boolean subposets, we know that $\rho \in \text{BOOL}(\tau)$ for some noncrossing tree τ on Q with convex geodesics. If the image $m(\tau)$ is a noncrossing tree on P with convex geodesics, then π belongs to the maximal Boolean subposet $\text{BOOL}(m(\tau))$ and we are done.

Suppose instead that $m(\tau)$ either fails to have convex geodesics or fails to be noncrossing. As a convenient shorthand, let z_- and z_+ be the elements of Q which

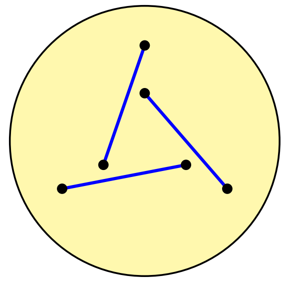

FIGURE 9. This noncrossing partition with vertex set P cannot be expressed as $\textsc{Part}(\mu)$ where μ is a subforest of a noncrossing tree with convex geodesics on P, so it does not belong to any maximal Boolean subposet of $\mathrm{NC}(P)$.

appear immediately before and immediately after $z_{i,0}$ in the counterclockwise cyclic order on Q. Concretely, $z_- = z_{i-1,c_{i-1}}$ and z_+ is either $z_{i,1}$ if $c_i > 0$ or $z_{i+1,0}$ if $c_i = 0$. If $m(\tau)$ fails to have convex geodesics, then this means by definition that $z_{i,0}$ does not lie on the geodesic from $z_{i-1,0}$ to $z_{i+1,0}$, which in particular means that $z_{i,0}$ does not lie on the geodesic in τ from z_- to z_+. If $m(\tau)$ fails to be noncrossing, then τ includes an edge between the sides labeled $i-1$ and i in $\textsc{Conv}(Q)$ which does not include $z_{i,0}$, and we are again in the preceding case: $z_{i,0}$ does not lie on the geodesic in τ from z_- to z_+.

Let us relabel the vertices in the geodesic in τ from z_- to z_+ in the following manner: $z_- = v_0, v_1, \ldots, v_h, v_{h+1} = z_+$. Then $z_{i,0}$ is the endpoint of a unique edge in τ, and the other endpoint is some v_j on the aforementioned geodesic—see Figure 8 for an illustration. If v_j is in the same block of ρ as $z_{i,0}$, then by Lemma 3.8, we can slide the edge connecting v_j to v_{j-1} (or the edge connecting v_j to v_{j+1}) along the edge connecting v_j to $z_{i,0}$ to obtain a noncrossing tree τ' with convex geodesics on Q such that $\rho \in \textsc{Bool}(\tau')$ and $m(\tau')$ is a noncrossing tree with convex geodesics on P.

Now suppose that v_j and $z_{i,0}$ do not share a block in ρ, and observe that this implies that $z_{i,0}$ must be a singleton in ρ, else τ contains an edge between two points in the block containing $z_{i,0}$, and this necessarily crosses an edge in the geodesic in τ from z_- to z_+, contradicting the assumption that τ is noncrossing. Since we know that $\pi = m(\rho)$ is a noncrossing partition of the configuration $P = m(Q)$, it can't be the case that a point on side $i-1$ of $\textsc{Conv}(P)$ shares a block with a point on side i while $z_{i,0}$ is a singleton block. It then follows that there is some v_l of minimal distance from v_j such that v_l and v_j are not in the same block, but every point on the geodesic between those two is in the same block as v_j. Then we can slide the edge between v_j and $z_{i,0}$ along the geodesic until we reach the vertex before v_l, which must not lie on side $i-1$ or side i, at which point we are in the setting of the previous case, and the proof is complete. □

It is worth noting that there are some configurations P (which are not hull configurations) such that $\mathrm{NC}(P)$ is not a union of maximal Boolean subposets. See Figure 9 for an example and observe that in this instance, $\mathrm{NC}(P)$ is not even graded [CDHM26, Section 4].

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