NONCROSSING PARTITIONS FROM CONES AND SEMICIRCLES

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Abstract. For each finite configuration of distinct points in the plane, there is an associated lattice of noncrossing partitions. When these points form the vertices of a convex polygon, the result is the classical noncrossing partition lattice, which is enumerated by the Catalan numbers and satisfies many other useful properties. In this article, we examine three variations of this lattice which arise when the starting configuration is allowed to have points on the sides of a convex polygon rather than just the vertex set.

1. Introduction

Let P be a finite set of points in the plane. A partition of P into blocks is *noncrossing* if the convex hulls of the blocks are pairwise disjoint, and the set of all such partitions—denoted $\mathrm{NC}(P)$—is partially ordered by refinement, forming a subposet of the full lattice of partitions of P. When P is the vertex set of a convex n-gon, the poset $\mathrm{NC}(P)$ is the classical *noncrossing partition lattice* $\mathrm{NC}(n)$, which was introduced by Kreweras [6] and has since taken on an important role in combinatorics, geometric group theory, and other fields. See [1] and [7] for surveys. At another extreme, if P is a configuration of n points which are all collinear, then $\mathrm{NC}(P)$ is isomorphic to the Boolean lattice of height $n-1$, i.e. the poset of all subsets of $\{1, \ldots, n-1\}$ under inclusion.

The Boolean lattice $\mathrm{Bool}(n)$ and the classical lattice of noncrossing partitions $\mathrm{NC}(n)$ satisfy a number of interesting poset properties. To name only a few, both of these lattices are bounded and graded, both are self-dual and admit symmetric chain decompositions [5, 9], and they are enumerated by interesting integer sequences (powers of 2 and the *Catalan numbers*, respectively). Both lattices are also of interest in the study of Artin groups due to their relationship to Garside structures for free abelian groups and braid groups, respectively [2, 3].

The appearance of these two important posets from the same construction suggests studying a natural generalization which incorporates both examples.

Question 1.1. *If P is a finite set of points which lie in the boundary of a convex polygon (rather than merely its vertices), then what can be said about $\mathrm{NC}(P)$?*

It is straightforward to see that $\mathrm{NC}(P)$ is a bounded lattice for any choice of configuration P [4, Proposition 2.3], but other properties vary. For example, there are large classes of configurations P for which $\mathrm{NC}(P)$ is also counted by the Catalan numbers (without being isomorphic to the classical lattice of noncrossing

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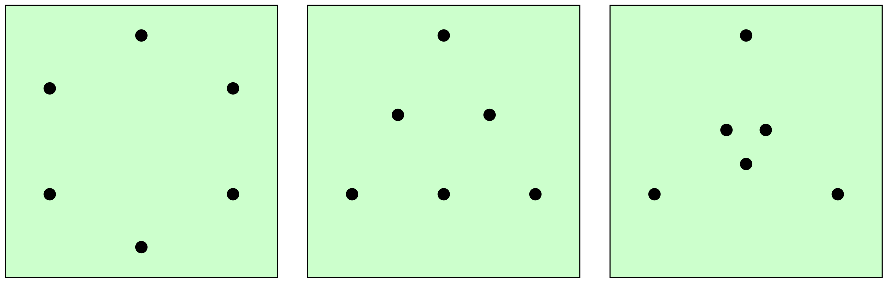

FIGURE 1. The lattice of noncrossing partitions for the leftmost configuration is both graded and rank-symmetric, whereas the lattice of noncrossing partitions for the middle configuration is graded but not rank-symmetric, and the lattice of noncrossing partitions for the rightmost configuration is not even graded.

partitions), yet there are other configurations for which the corresponding lattice of noncrossing partitions is not rank-symmetric (or even graded)—see Figure 1. Our first theorem demonstrates that the configurations considered in Question 1.1 always produce graded lattices.

Theorem A (Theorem 2.3)**.** *If P is a finite subset of $\mathbb{C}$ which lies on the boundary of a convex polygon, then* $\mathrm{NC}(P)$ *is graded.*

For other poset properties, we investigate the noncrossing partitions for three special types of configurations which arise in the setting of Question 1.1.

Definition 1.2. For each pair of nonnegative integers m and n, we define three different types of configurations in the plane.

(1) An *open cone configuration* $U_{m,n}$ consists of $m+n$ points on the boundary of an affine convex cone with m points on one bounding ray, n points on the other ray, and no point on their intersection.
(2) A *closed cone configuration* $V_{m,n}$ consists of $m+n+1$ points on the boundary of an affine convex cone with m points on one bounding ray, n points on the other ray, and one point on their intersection. Removing the intersection point turns $V_{m,n}$ into $U_{m,n}$.
(3) A *semicircular configuration* $S_{m,n}$ consists of $m+n+2$ points on a semicircle, with $m+2$ points on the flat side (including the corners) and n points on the round side (not including the corners).

See Figure 2 for illustrations. Note that the isomorphism type of the noncrossing partition lattice for each type of configuration is determined only by m and n, so it is well-defined to write e.g. $\mathrm{NC}(U_{m,n})$ without specifying the coordinates of the configuration. Also, note that if P consists of m points on one line and n points on a distinct parallel line, then $\mathrm{NC}(P)$ is isomorphic to $\mathrm{NC}(U_{m,n})$.

Our next main theorem addresses symmetric chain decompositions in noncrossing partition lattices for each of the three configuration types above.

Theorem B (Theorems 3.2 and 4.2)**.** *For all m and n, the bounded graded lattices* $\mathrm{NC}(U_{m,n})$*,* $\mathrm{NC}(V_{m,n})$ *and* $\mathrm{NC}(S_{m,n})$ *admit symmetric chain decompositions. Consequently, all three lattices are rank-symmetric.*

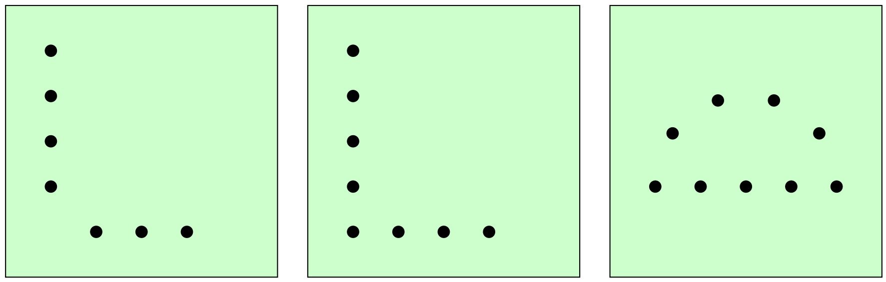

FIGURE 2. From left to right: the open cone configuration $U_{3,4}$, the closed cone configuration $V_{3,4}$, and the semicircular configuration $S_{3,4}$.

It is worth noting that despite being rank-symmetric, these posets are generally not self-dual. Combined with the observations presented in [4], it would seem that self-duality is a relatively rare property among noncrossing partition lattices from configurations.

Our final main theorem provides multivariate generating functions which enumerate each of the three lattices.

Theorem C (Theorems 3.3, 3.4 and 4.3)**.** *Let $u_{m,n}$, $v_{m,n}$ and $s_{m,n}$ denote the sizes of $\mathrm{NC}(U_{m,n})$, $\mathrm{NC}(V_{m,n})$ and $\mathrm{NC}(S_{m,n})$ respectively. Then the corresponding multivariate generating functions are as follows:*

(1) $U(x,y) = \sum_{m,n\geq 0} u_{m,n}x^m y^n = \dfrac{x+y-2xy}{1-2x-2y+3xy}$;

(2) $V(x,y) = \sum_{m,n\geq 0} v_{m,n}x^m y^n = \dfrac{1}{1-2x-2y+3xy}$;

(3) $S(x,y) = \sum_{m,n\geq 0} s_{m,n}x^m y^n = \left(\dfrac{C(y)-1-y}{y^2}\right)\left(\dfrac{1}{1-x(1+C(y))}\right)$,

where $C(y) = (1-\sqrt{1-4y})/2y$ is the generating function for the Catalan numbers.

It is worth noting that, in a sense, the generating function for counting semicircular noncrossing partitions interpolates between $C(y)$, the generating function which enumerates classical noncrossing partitions, and $1/(1-2x)$, the generating function which enumerates noncrossing partitions of collinear points.

The article is structured as follows. In Section 2, we introduce some preliminary material on posets and noncrossing partitions before proving Theorem A and investigating a particular example which is common to all three configuration types. The proofs of Theorems B and C are divided across Sections 3 and 4, which examine (open and closed) cone configurations and semicircular configurations, respectively.

2. Noncrossing partitions from configurations

In this section we recall some background on posets and noncrossing partitions. See [4, Section 2] for more detail.

Definition 2.1. Let S be a finite set. A *partition* of S is a collection of pairwise disjoint subsets of S (called *blocks*) whose union is S. The set of all partitions of

S, denoted $\Pi(S)$, is partially ordered under refinement, i.e. $\pi \leq \mu$ in $\Pi(S)$ if every block of μ is a union of blocks in π. Moreover, $\Pi(S)$ has the following properties.

- $\Pi(S)$ is *bounded*: it has a unique minimum $\hat{0}$ and a unique maximum $\hat{1}$.
- $\Pi(S)$ is a *lattice*: each pair of elements π and μ has a unique meet (greatest lower bound) $\pi \wedge \mu$ and a unique join (least upper bound) $\pi \vee \mu$.
- $\Pi(S)$ is *graded*: if $bl(\pi)$ is the number of blocks in π, then the function $\rho\colon \Pi(S) \to \mathbb{N}$ given by $\rho(\pi) = |S| - bl(\pi)$ is a *rank function* for $\Pi(S)$, which means that $\rho(\pi) \leq \rho(\mu)$ when $\pi \leq \mu$ and $\rho(\pi) + 1 = \rho(\mu)$ when $\rho < \mu$ and there is no $\eta \in \Pi(S)$ with $\rho < \eta < \mu$.

If $|S| = n$, then $\hat{0}$ has rank 0 and $\hat{1}$ has rank $n-1$. The *atoms* and *coatoms* of $\Pi(S)$ are the elements of rank 1 and $n-2$, respectively.

When the set S has geometric meaning, we can restrict $\Pi(S)$ to an interesting subposet.

Definition 2.2. Let P be a finite subset of $\mathbb{C}$ (referred to here as a *configuration*). For each subset $A \subseteq P$, its *convex hull* $\textsc{Conv}(A)$ is the smallest convex set in $\mathbb{C}$ which contains P. Since P is assumed to be finite, $\textsc{Conv}(A)$ is necessarily a convex polygon with up to $|A|$ vertices. A partition of the configuration P is *noncrossing* if the convex hulls of its blocks are pairwise disjoint, and the set of all noncrossing partitions $\mathrm{NC}(P)$ is a subposet of the partition lattice $\Pi(P)$.

The properties held by the poset $\mathrm{NC}(P)$ can vary dramatically based on the choice of P. For example, $\mathrm{NC}(P)$ is always a bounded lattice, but it is not always graded [4]. In the cases presented in this article, however, we do not need to worry about these exceptions.

Theorem 2.3. *If P is a finite subset of $\mathbb{C}$ which lies on the boundary of a convex polygon, then $\mathrm{NC}(P)$ is graded.*

Proof. Let ρ be the rank function for $\Pi(P)$ described in Definition 2.1 and let $\pi < \mu$ be a covering relation in $\mathrm{NC}(P)$, which is to say that π and μ are noncrossing partitions of P with the property that there is no $\eta \in \mathrm{NC}(P)$ with $\pi < \eta < \mu$. By definition of the refinement order, this means that μ is obtained by removing some blocks $R_1, \ldots, R_k$ from π and replacing them with the union of those blocks, where $k \geq 2$. If we can show that $k = 2$, then we know that $\rho(\pi) + 1 = \rho(\mu)$ and therefore ρ is a rank function for $\mathrm{NC}(P)$.

Consider the removed block R_1. One of the sides of the convex hull $\textsc{Conv}(R_1)$ must belong to a line L such that some other $\textsc{Conv}(R_i)$ lies on the opposite side, i.e. so that L separates the interiors of $\textsc{Conv}(R_1)$ and $\textsc{Conv}(R_i)$. Moreover, because all of P lies on the boundary of a convex polygon, the convex hull of each other block lies entirely within one of the half-planes bounded by L. Without loss of generality, suppose that the interiors of the convex hulls of $R_1, \ldots, R_{i-1}$ lie on one side of L and that the interiors of the convex hulls of $R_i, \ldots, R_k$ lie on the other.

Now, define $\eta \in \mathrm{NC}(P)$ from π by replacing $R_1, \ldots, R_{i-1}$ with their union and replacing $R_i, \ldots, R_k$ with their union. It is then clear that $\pi \leq \eta < \mu$, with equality if and only if $k = 2$. Since we assumed that $\pi < \mu$ was a covering relation, we therefore must have $k = 2$ and the proof is complete. ☐

Other potential properties of $\mathrm{NC}(P)$ require a bit of introduction. First, $\mathrm{NC}(P)$ is *self-dual* if there is a bijection $f\colon \mathrm{NC}(P) \to \mathrm{NC}(P)$ such that $\pi \leq \mu$ if and only

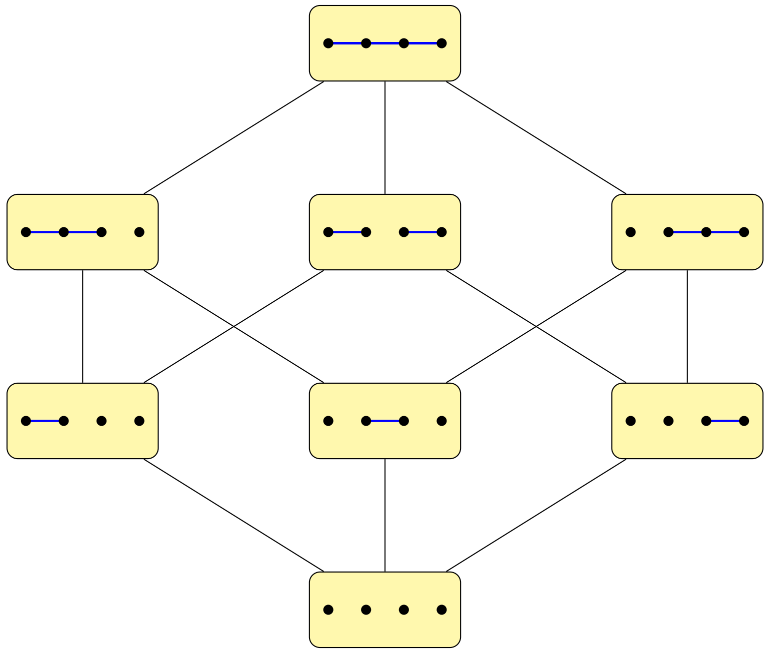

FIGURE 3. The lattice of noncrossing partitions $\mathrm{NC}(P_4)$ defined in Example 2.4 is isomorphic to the Boolean lattice $\textsc{Bool}(3)$.

if $f(\mu) \leq f(\pi)$. If $\mathrm{NC}(P)$ is graded and the number of elements with rank k is equal to the number of elements with rank $n-k-1$ for all $k \in \{0, \ldots, n-1\}$, then $\mathrm{NC}(P)$ is *rank-symmetric*. Similarly, if A is a subposet of $\mathrm{NC}(P)$ such that the number of elements of rank k in A is equal to the number of elements of rank $n-k-1$ in A (both with respect to the rank function of $\mathrm{NC}(P)$), then we say that A is *centered*. Given $\pi \leq \mu$ in $\mathrm{NC}(P)$, the *interval* $[\pi, \mu]$ is defined to be the subposet $\{\eta \in \mathrm{NC}(P) \mid \pi \leq \eta \leq \mu\}$. Also, a totally ordered subposet of $\mathrm{NC}(P)$ is called a *chain*; a chain is *maximal* if it is not properly contained in another chain and *saturated* if whenever $\pi < \eta < \mu$ in $\mathrm{NC}(P)$ with π and μ in the chain, then η must belong to the chain as well. Finally, a graded poset admits a *symmetric chain decomposition* if its elements can be expressed as the disjoint union of centered saturated chains.

The two most natural examples of $\mathrm{NC}(P)$ arise when the configuration P either lies on a line or on a circle, and each of these produces a familiar lattice with useful properties.

Example 2.4. If P_n consists of points $p_1, \ldots, p_n$ which are arranged in order along a line, then each noncrossing partition of P_n is determined by a choice for each $i \in \{1, \ldots, n-1\}$ of whether p_i and p_{i+1} belong to the same block. This provides an isomorphism between $\mathrm{NC}(P_n)$ and the poset of all subsets of $\{1, \ldots, n-1\}$ under inclusion, known as the *Boolean lattice* $\textsc{Bool}(n-1)$. See Figure 3 for an illustration. Note in particular that $\mathrm{NC}(P_n)$ is graded and self-dual (and therefore rank-symmetric). Moreover, the size of $\mathrm{NC}(P_n)$ is 2^{n-1}, and the generating

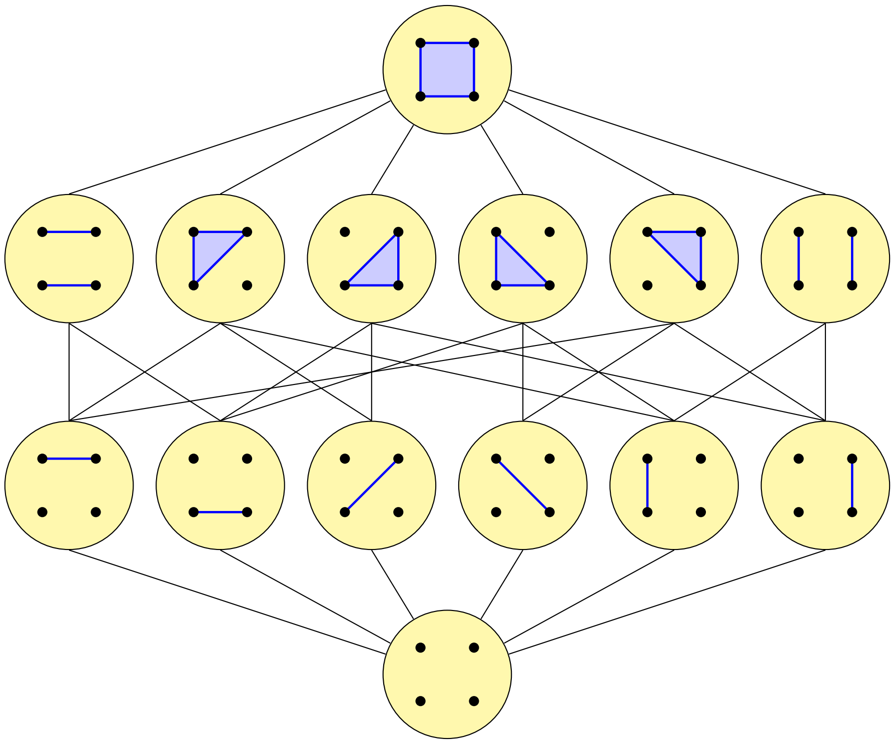

FIGURE 4. The classical lattice of noncrossing partitions NC(4)

function for this sequence is

$$B(x) = \sum_{n \geq 0} 2^{n-1} x^n = \frac{x}{1-2x}.$$

Finally, the Boolean lattice (and more generally any product of chains) admits a symmetric chain decomposition [5].

Example 2.5. If the configuration Q_n is the vertex set of a convex n-gon, then $\mathrm{NC}(Q_n)$ is the classical *lattice of noncrossing partitions* $\mathrm{NC}(n)$. This lattice is graded, rank-symmetric, and counted by the n-th *Catalan number* $C_n = \frac{1}{n+1}\binom{2n}{n}$ [6]; it is also self-dual and admits a symmetric chain decomposition [9]. Moreover, the generating function for this sequence is

$$C(x) = \sum_{n \geq 0} C_n x^n = \frac{1 - \sqrt{1-4x}}{2x}.$$

See Figure 4 for an illustration of NC(4).

We close this section with an example which belongs to all three of the configuration types introduced in Definition 1.2. Let T_n denote a configuration of $n+1$ points $x_1, \dots, x_n, y$ in $\mathbb{C}$ in which $x_1, \dots, x_n$ lie in order on a common line and the remaining point y does not. To address the recursive structure of $\mathrm{NC}(T_n)$, we will need a technical lemma.

Lemma 2.6. *Let $P = \{p_1, \ldots, p_n\}$ be a configuration in $\mathbb{C}$ which lies on the boundary of a convex polygon in counterclockwise order, and suppose that p_n and p_1 are vertices of the polygon.*

(1) The subposet of $\mathrm{NC}(P)$ consisting of all partitions where p_{n-1} and p_n share a block is isomorphic to $\mathrm{NC}(P - \{p_n\})$.
(2) The subposet of $\mathrm{NC}(P)$ consisting of all partitions where either p_{n-1} and p_n share a block or p_n is a singleton is isomorphic to the direct product $\mathrm{NC}(P - \{p_n\}) \times \mathrm{NC}(\{p_{n-1}, p_n\})$.

Proof. For the first claim, define $\alpha = \{\{p_1\}, \ldots, \{p_{n-2}\}, \{p_{n-1}, p_n\}\}$ and observe that the described subposet is the interval $[\alpha, \hat{1}]$ in $\mathrm{NC}(P)$. For each partition π in $\mathrm{NC}(P - \{p_n\})$, define $f(\pi)$ simply by adding p_n to the block containing p_{n-1}. Since p_n and p_1 are vertices of the polygon containing P in its boundary, we can see that $f(\pi)$ is also noncrossing, and in particular an element of $[\alpha, \hat{1}]$. The fact that f is an order-preserving bijection with order-preserving inverse then follows immediately from the definition, so the first claim is proved.

Next, define $\beta = \{\{p_1, \ldots, p_{n-1}\}, \{p_n\}\}$ and observe that $\beta \in \mathrm{NC}(P)$ since we assumed that p_n is a vertex of the polygon. Then the subposet described in the second claim is the disjoint union of the intervals $[\alpha, \hat{1}]$ and $[\hat{0}, \beta]$. Since p_n is not in the convex hull of $\{p_1, \ldots, p_{n-1}\}$, we know that $[\hat{0}, \beta]$ is isomorphic to $\mathrm{NC}(P - \{p_n\})$, which we just showed is also isomorphic to $[\alpha, \hat{1}]$. Finally, observe that each partition in $[\alpha, \hat{1}]$ covers a unique element of $[\hat{0}, \beta]$ which is obtained by removing p_n from its block and making it into a singleton. Therefore, the subposet $[\alpha, \hat{1}] \cup [\hat{0}, \beta]$ is the direct product of $\mathrm{NC}(P - \{p_n\})$ and $\mathrm{NC}(\{p_{n-1}, p_n\})$. □

We are now ready to identify some recursive structure for $\mathrm{NC}(T_n)$. See Figure 5 for an illustration.

Lemma 2.7. *Let $n \geq 2$, let A be the subposet of $\mathrm{NC}(T_n)$ which consists of all partitions in which x_n is either a singleton or in a block with x_{n-1}, and let B be the subposet of $\mathrm{NC}(T_n)$ consisting of all partitions in which $\{x_n, y\}$ is a block. Then*

(1) A is isomorphic to $\mathrm{NC}(T_{n-1}) \times \mathrm{Bool}(1)$;
(2) B is isomorphic to $\mathrm{Bool}(n-2)$.

Moreover, $\mathrm{NC}(T_n)$ is the disjoint union of A and B.

Proof. The first claim follows from Lemma 2.6. The second claim is straightforward since the convex hull of $\{x_n, y\}$ is disjoint from the convex hull of $\{x_1, \ldots, x_{n-1}\}$, and $\mathrm{NC}(\{x_1, \ldots, x_{n-1}\})$ is isomorphic to $\mathrm{Bool}(n-2)$. For the final claim, note that the arrangement of the configuration T_n tells us that x_n must either be in a block by itself, be in a block with y and nothing else, or share a block with x_{n-1} (and possibly other points). So the claim that $\mathrm{NC}(T_n)$ is the disjoint union of A and B is immediate and the proof is complete. □

Next, we can address the presence of a symmetric chain decomposition. The following observation will be useful at several points in the rest of the article.

Remark 2.8. Suppose $\mathrm{NC}(P)$ is graded with rank function ρ, and suppose that $\mathrm{NC}(P)$ is the disjoint union of subposets $A_1, \ldots, A_k$ such that each A_i is a bounded poset with minimum element $\hat{0}_i$ and maximum element $\hat{1}_i$. If

(1) each A_i is a union of intervals in $\mathrm{NC}(P)$,
(2) each A_i admits a symmetric chain decomposition, and

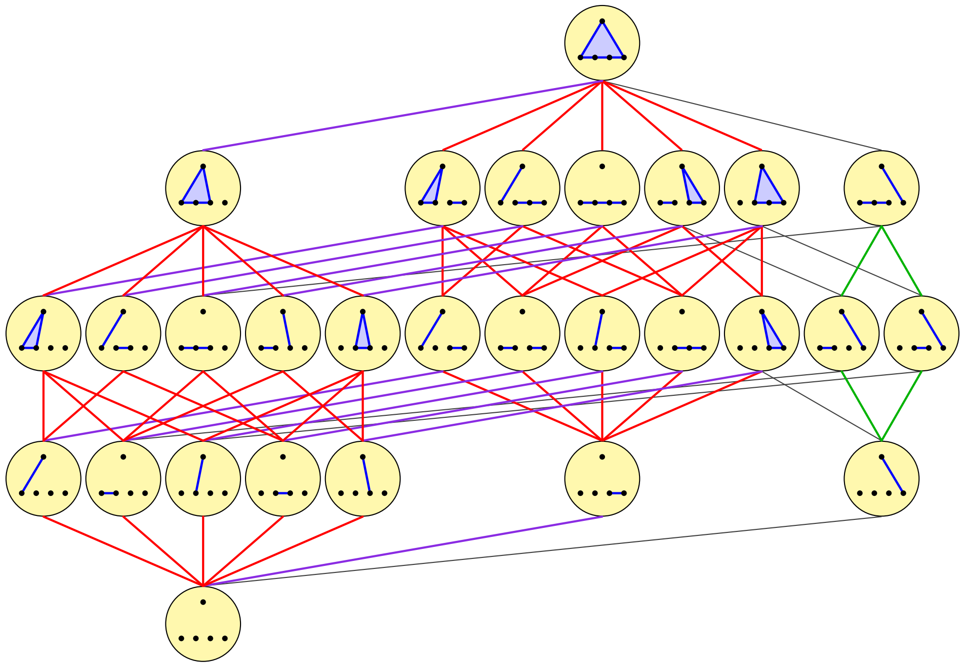

FIGURE 5. The noncrossing partition lattice $\mathrm{NC}(T_4)$. The edges have been colored so that the red edges and purple edges illustrate the subposet A defined in Lemma 2.7 and the green edges illustrate the subposet B.

(3) $\rho(\hat{0})+\rho(\hat{1})=\rho(\hat{0}_i)+\rho(\hat{1}_i)$ for all $i \in \{1,\ldots,k\}$,

then $\mathrm{NC}(P)$ is a union of centered subposets with symmetric chain decompositions which can be combined to form a symmetric chain decomposition for $\mathrm{NC}(P)$.

Proposition 2.9. $\mathrm{NC}(T_n)$ *admits a symmetric chain decomposition.*

Proof. First, we know from Theorem 2.3 that $\mathrm{NC}(T_n)$ is graded. As base cases, note that $\mathrm{NC}(T_0)$ and $\mathrm{NC}(T_1)$ consist of only one element and two elements respectively, and they trivially have symmetric chain decompositions as they each consist of a single chain. Proceeding by induction on n, suppose that $\mathrm{NC}(T_{n-1})$ has a symmetric chain decomposition; in particular, this means that $\mathrm{NC}(T_{n-1})$ is rank-symmetric. For $n \geq 2$, we know by Lemma 2.7 that $\mathrm{NC}(T_n)$ is the disjoint union of subposets isomorphic to $\mathrm{NC}(T_{n-1}) \times \mathrm{BOOL}(1)$ and $\mathrm{BOOL}(n-2)$. Products of posets with symmetric chain decompositions have symmetric chain decompositions [5], so the subposets A and B meet the criteria for Remark 2.8. Therefore, $\mathrm{NC}(T_n)$ has a symmetric chain decomposition as well, and we are done. □

Proposition 2.10. *If* $t_n = |\mathrm{NC}(T_n)|$, *then the associated generating function is*

$$T(x)=\sum_{n\geq 0} t_n x^n = \frac{(1-x)^2}{(1-2x)^2}.$$

Proof. When $n \geq 2$, we know that t_n satisfies the recursive formula

$$t_n = 2t_{n-1} + 2^{n-2}$$

by Lemma 2.7, and the sequence begins with the terms $1, 2, 5, 12, 28, 64, \ldots$ for $n \geq 0$. Therefore, we have

$$\begin{aligned}
T(x) &= \sum_{n\geq 0} t_n x^n \\
&= 1 + 2x + \sum_{n\geq 2} t_n x^n \\
&= 1 + 2x + 2\sum_{n\geq 2} t_{n-1}x^n + \sum_{n\geq 2} 2^{n-2}x^n \\
&= 1 + 2x + 2x\sum_{n\geq 1} t_n x^n + x^2\sum_{n\geq 0} 2^n x^n \\
&= 1 + 2x + 2x(T(x)-1) + x^2\frac{1}{1-2x} \\
T(x)(1-2x) &= 1 + x^2\frac{1}{1-2x} \\
T(x)(1-2x) &= \frac{1-2x+x^2}{1-2x} \\
T(x) &= \frac{(1-x)^2}{(1-2x)^2}
\end{aligned}$$

and we are done. □

As a final note, we observe that there is a closed form for the recursive formula given in the proof of Proposition 2.10: $t_n = (n+3)2^{n-2}$ when $n \geq 2$. To see this, observe as a base case that $t_2 = 5$, then assume that for some $n \geq 2$, the formula works for t_n; then we have

$$\begin{aligned}
t_{n+1} &= 2t_n + 2^{n-1} \\
&= 2(n+3)2^{n-2} + 2^{n-1} \\
&= (n+3)2^{n-1} + 2^{n-1} \\
&= (n+4)2^{n-1}
\end{aligned}$$

as desired.

3. Cone configurations

Recall the definitions of open cone configurations $U_{m,n}$ and closed cone configurations $V_{m,n}$ from Definition 1.2. In this section we will show that the corresponding lattices $\mathrm{NC}(U_{m,n})$ and $\mathrm{NC}(V_{m,n})$ admit symmetric chain decompositions and provide multivariate generating functions which count the number of elements in each. For the remainder of this section, assume that $U_{m,n}$ consists of m points $x_1, \ldots, x_m$ in order on the positive real axis and n points $y_1, \ldots, y_n$ in order on the positive imaginary axis in $\mathbb{C}$, and let $V_{m,n}$ be the same configuration with the additional point z at the origin.

Lemma 3.1. *Let $m \geq 2$ and $n \geq 1$. Define A to be the subposet of $\mathrm{NC}(U_{m,n})$ which consists of all partitions in which x_m is either a singleton or in a block with x_{m-1}. For each $k \in \{1, \ldots, n\}$, define B_k to be the subposet of $\mathrm{NC}(U_{m,n})$ consisting of all partitions in which x_m shares a block with y_k but not $x_{m-1}, y_1, \ldots, y_{k-1}$. Then*

(1) *A is isomorphic to $\mathrm{NC}(U_{m-1,n}) \times \mathrm{Bool}(1)$;*

$|\mathrm{NC}(U_{m,n})|$

$m\backslash n$	0	1	2	3	4
0	0	1	2	4	8
1	1	2	5	12	28
2	2	5	14	37	94
3	4	12	37	106	289
4	8	28	94	289	838

$|\mathrm{NC}(V_{m,n})|$

$m\backslash n$	0	1	2	3	4
0	1	2	4	8	16
1	2	5	12	28	64
2	4	12	33	86	216
3	8	28	86	245	664
4	16	64	216	664	1921

TABLE 1. Sizes of $\mathrm{NC}(U_{m,n})$ and $\mathrm{NC}(V_{m,n})$ for small values of m and n

(2) B_k *is isomorphic to* $\mathrm{NC}(U_{m-1,k-1}) \times \mathrm{Bool}(n-k)$

Moreover, $\mathrm{NC}(U_{m,n})$ *is the disjoint union of* $A, B_1, \ldots, B_n$*.*

Proof. The first claim follows from Lemma 2.6. For the second claim, we know by Lemma 2.6 that B_k is isomorphic to $\mathrm{NC}(U_{m-1,k-1}) \times \mathrm{NC}(U_{0,n-k+1})$, and we know that the second factor is isomorphic to $\mathrm{Bool}(n-k)$. Finally, we know that in each partition of $U_{m,n}$, the point x_m must either be a singleton or share a block with x_{m-1} or one of $y_1, \ldots, y_n$, so $\mathrm{NC}(U_{m,n})$ is the union of $A, B_1, \ldots, B_n$, and it is clear from the definition that these are disjoint. □

Note that one may replace $U_{m,n}$ in the proof above with $V_{m,n}$ to obtain an identical result, as the additional point z does not require any extra posets for the decomposition. Thus, we can use Lemma 3.1 for both $\mathrm{NC}(U_{m,n})$ and $\mathrm{NC}(V_{m,n})$. As a first application, we use this recursive structure to show that both lattices have symmetric chain decompositions.

Theorem 3.2. $\mathrm{NC}(U_{m,n})$ *and* $\mathrm{NC}(V_{m,n})$ *admit symmetric chain decompositions.*

Proof. First, observe that $\mathrm{NC}(U_{0,n})$ and $\mathrm{NC}(U_{m,0})$ are Boolean lattices and therefore admit symmetric chain decompositions. We also have $\mathrm{NC}(U_{1,n}) \cong \mathrm{NC}(T_n)$, which has a symmetric chain decomposition by Proposition 2.9.

Proceeding by induction on both m and n, suppose that $\mathrm{NC}(U_{a,b})$ has a symmetric chain decomposition when $a < m$ and $b \leq n$. When $m \geq 2$ and $n \geq 1$, Lemma 3.1 tells us that $\mathrm{NC}(U_{m,n})$ is the disjoint union of subposets which are isomorphic to $\mathrm{NC}(U_{m-1,n}) \times \mathrm{Bool}(1)$ and $\mathrm{NC}(U_{m-1,k-1}) \times \mathrm{Bool}(n-k)$ for $k \in \{1, \ldots, n\}$. Combining the inductive hypothesis with the fact that Boolean lattices have symmetric chain decompositions, we know by Remark 2.8 that $\mathrm{NC}(U_{m,n})$ is the union of disjoint centered subposets with symmetric chain decompositions, and therefore $\mathrm{NC}(U_{m,n})$ itself admits a symmetric chain decomposition. The same result for $\mathrm{NC}(V_{m,n})$ follows analogously. □

To close this section, we examine the sizes of $\mathrm{NC}(U_{m,n})$ and $\mathrm{NC}(V_{m,n})$. See Table 1 for values with small m and n. Note that the size of $\mathrm{NC}(U_{m,n})$ matches the OEIS sequence A035002 and the size of $\mathrm{NC}(V_{m,n})$ matches the sequence A341867 [8]. More concretely, we now provide multivariate generating functions for the sizes of $\mathrm{NC}(U_{m,n})$ and $\mathrm{NC}(V_{m,n})$.

Theorem 3.3. *If $u_{m,n}$ is the size of* $\mathrm{NC}(U_{m,n})$*, then the associated multivariate generating function is*

$$U(x,y)=\sum_{m,n\geq 0}u_{m,n}x^m y^n=\frac{x+y-2xy}{1-2x-2y+3xy}.$$

Proof. As a first step, we have

$$\begin{aligned}U(x,y)&=\sum_{\substack{m\geq 0\\ n\geq 0}}u_{m,n}x^m y^n\\ &=\sum_{n\geq 0}u_{0,n}y^n+\sum_{n\geq 0}u_{1,n}xy^n+\sum_{m\geq 2}u_{m,0}x^m+\sum_{\substack{m\geq 2\\ n\geq 1}}u_{m,n}x^m y^n.\end{aligned}$$

The first three sums are straightforward to deal with. The first is

$$\sum_{n\geq 0}u_{0,n}y^n=\sum_{n\geq 1}2^{n-1}y^n=\frac{y}{1-2y}$$

by Example 2.4, the second is

$$\sum_{n\geq 0}u_{1,n}xy^n=x\sum_{n\geq 0}t_n y^n=x\frac{(1-y)^2}{(1-2y)^2}$$

by Proposition 2.10, and the third is

$$\sum_{m\geq 2}u_{m,0}x^m=\sum_{m\geq 2}2^{m-1}x^m=\frac{2x^2}{1-2x}$$

by Example 2.4 again. Combining these three sums, we obtain

$$\sum_{n\geq 0}u_{0,n}y^n+\sum_{n\geq 0}u_{1,n}xy^n+\sum_{m\geq 2}u_{m,0}x^m=\frac{y}{1-2y}+x\frac{(1-y)^2}{(1-2y)^2}+\frac{2x^2}{1-2x}.$$

As for the fourth sum, we know by Lemma 3.1 that

$$u_{m,n}=2u_{m-1,n}+\sum_{k=1}^{n}u_{m-1,k-1}2^{n-k}$$

when $m \geq 2$ and $n \geq 1$. Applying this recursive formula, we can write the fourth sum as follows:

$$\begin{aligned}
\sum_{\substack{m\geq 2\\ n\geq 1}} u_{m,n}x^m y^n &= 2\sum_{\substack{m\geq 2\\ n\geq 1}} u_{m-1,n}x^m y^n + \sum_{\substack{m\geq 2\\ n\geq 1}}\sum_{k=1}^{n} u_{m-1,k-1}2^{n-k}x^m y^n \\
&= 2x\sum_{\substack{m\geq 1\\ n\geq 1}} u_{m,n}x^m y^n + xy\sum_{\substack{m\geq 1\\ n\geq 1}}\sum_{k=0}^{n-1} u_{m,k}2^{n-k-1}x^m y^{n-1} \\
&= 2x\sum_{\substack{m\geq 1\\ n\geq 1}} u_{m,n}x^m y^n + xy\left(\sum_{\substack{m\geq 1\\ n\geq 0}} u_{m,n}x^m y^n\right)\left(\sum_{n\geq 0} 2^n y^n\right) \\
&= 2x\left(U(x,y) - \frac{x}{1-2x} - \frac{y}{1-2y}\right) + \left(U(x,y) - \frac{y}{1-2y}\right)\frac{xy}{1-2y} \\
&= U(x,y)\left(2x + \frac{xy}{1-2y}\right) - \frac{2x^2}{1-2x} - \frac{2xy}{1-2y} - \frac{xy^2}{(1-2y)^2}.
\end{aligned}$$

Putting it all together, we observe that two terms immediately cancel and we have

$$U(x,y) = U(x,y)\frac{2x-3xy}{1-2y} + \frac{y}{1-2y} + x\frac{(1-y)^2}{(1-2y)^2} - \frac{2xy}{1-2y} - \frac{xy^2}{(1-2y)^2},$$

which can be rearranged to get

$$\begin{aligned}
U(x,y)\left(1 - \frac{2x-3xy}{1-2y}\right) &= \frac{y-2xy}{1-2y} + \frac{x(1-y)^2 - xy^2}{(1-2y)^2} \\
U(x,y)\frac{1-2x-2y+3xy}{1-2y} &= \frac{y(1-2x)}{1-2y} + \frac{x(1-2y)}{(1-2y)^2} \\
U(x,y)(1-2x-2y+3xy) &= x+y-2xy \\
U(x,y) &= \frac{x+y-2xy}{1-2x-2y+3xy}
\end{aligned}$$

and we are done. □

Theorem 3.4. *If $v_{m,n}$ is the size of $\mathrm{NC}(V_{m,n})$, then the associated multivariate generating function is*

$$V(x,y) = \sum_{m,n\geq 0} v_{m,n}x^m y^n = \frac{1}{1-2x-2y+3xy}.$$

Proof. First, we have

$$\begin{aligned}
V(x,y) &= \sum_{\substack{m\geq 0\\ n\geq 0}} v_{m,n}x^m y^n \\
&= v_{0,0} + \sum_{n\geq 1} v_{0,n}y^n + \sum_{m\geq 1} v_{m,0}x^m + \sum_{\substack{m\geq 1\\ n\geq 1}} v_{m,n}x^m y^n \\
&= 1 + \frac{2y}{1-2y} + \frac{2x}{1-2x} + \sum_{\substack{m\geq 1\\ n\geq 1}} v_{m,n}x^m y^n. \qquad (1)
\end{aligned}$$

Recalling that $V_{m,n}$ is obtained from $U_{m,n}$ by including the additional corner point z, we observe that z is either a singleton or it shares a block with x_1, y_1, or both. When m and n are at least 1, we thus have the recursive enumeration formula:

$$v_{m,n} = u_{m,n} + v_{m-1,n} + v_{m,n-1} - v_{m-1,n-1}.$$

The first term on the righthand side corresponds to the case where z is a singleton. The second and third terms correspond to the case where z shares a block with x_1 or y_1 respectively, and we subtract the fourth term to account for the overlap, in which z shares a block with both x_1 and y_1.

Returning to the generating function, we can apply the recursive enumeration formula to write the remaining sum as

$$\sum_{\substack{m\geq 1\\ n\geq 1}} u_{m,n}x^m y^n + \sum_{\substack{m\geq 1\\ n\geq 1}} v_{m-1,n}x^m y^n + \sum_{\substack{m\geq 1\\ n\geq 1}} v_{m,n-1}x^m y^n - \sum_{\substack{m\geq 1\\ n\geq 1}} v_{m-1,n-1}x^m y^n,$$

which can be reindexed to provide

$$\sum_{\substack{m\geq 1\\ n\geq 1}} u_{m,n}x^m y^n + x\sum_{\substack{m\geq 0\\ n\geq 1}} v_{m,n}x^m y^n + y\sum_{\substack{m\geq 1\\ n\geq 0}} v_{m,n}x^m y^n - xy\sum_{\substack{m\geq 0\\ n\geq 0}} v_{m,n}x^m y^n.$$

We handle each of the four sums in turn:

- $\displaystyle\sum_{\substack{m\geq 1\\ n\geq 1}} u_{m,n}x^m y^n = U(x,y) - \frac{x}{1-2x} - \frac{y}{1-2y};$
- $\displaystyle x\sum_{\substack{m\geq 0\\ n\geq 1}} v_{m,n}x^m y^n = xV(x,y) - \frac{x}{1-2x};$
- $\displaystyle y\sum_{\substack{m\geq 1\\ n\geq 0}} v_{m,n}x^m y^n = yV(x,y) - \frac{y}{1-2y};$
- $\displaystyle xy\sum_{\substack{m\geq 0\\ n\geq 0}} v_{m,n}x^m y^n = xyV(x,y).$

Returning to Equation 1, we now have

$$\begin{aligned} V(x,y) &= 1 + U(x,y) + xV(x,y) + yV(x,y) - xyV(x,y)\\ V(x,y)(1-x-y+xy) &= 1 + \frac{x+y-2xy}{1-2x-2y+3xy}\\ V(x,y)(1-x-y+xy) &= \frac{1-x-y+xy}{1-2x-2y+3x}\\ V(x,y) &= \frac{1}{1-2x-2y+3xy}\end{aligned}$$

as desired. □

4. Semicircular configurations

In our final section, we will examine the noncrossing partition lattices for semicircular configurations, introduced in Definition 1.2. In particular, we show that these lattices admit symmetric chain decompositions and produce the associated multivariate generating functions. For the rest of this section, let $S_{m,n}$ denote a configuration of $m+2$ points $x_0,\ldots,x_{m+1}$ on the line segment from -1 to 1 in $\mathbb{C}$ and n points $y_1,\ldots,y_n$ on the upper half of the unit circle.

$|\mathrm{NC}(S_{m,n})|$

$m\backslash n$	0	1	2	3	4
0	2	5	14	42	132
1	4	12	37	118	387
2	8	28	94	317	1082
3	16	64	232	824	2921
4	32	144	560	2088	7674

TABLE 2. Sizes of $\mathrm{NC}(S_{m,n})$ for small values of m and n

As in the previous section, we begin with a lemma on the recursive structure for $\mathrm{NC}(S_{m,n})$. Note the similarities with Lemma 3.1.

Lemma 4.1. *Let $m \geq 1$ and $n \geq 1$. Define A to be the subposet of $\mathrm{NC}(S_{m,n})$ which consists of all partitions in which x_{m+1} is either a singleton or in a block with x_m. For each $k \in \{1, \ldots, n\}$, define B_k to be the subposet of $\mathrm{NC}(S_{m,n})$ consisting of all partitions in which x_{m+1} shares a block with y_k but not $x_m, y_1, \ldots, y_{k-1}$. Then*

(1) A is isomorphic to $\mathrm{NC}(S_{m-1,n}) \times \mathrm{BOOL}(1)$;
(2) B_k is isomorphic to $\mathrm{NC}(S_{m-1,k-1}) \times \mathrm{NC}(n-k+1)$.

Moreover, $\mathrm{NC}(S_{m,n})$ is the disjoint union of $A, B_1, \ldots, B_k$.

Proof. The first claim follows directly from Lemma 2.6. For the second, we know by Lemma 2.6 that B_k is isomorphic to $\mathrm{NC}(S_{m-1,k-1}) \times \mathrm{NC}(\{y_k, \ldots, y_n\})$, and since the points $y_k, \ldots, y_n$ are assumed to lie on the boundary of a circle, we know that $\mathrm{NC}(\{y_k, \ldots, y_n\})$ is isomorphic to the classical lattice of noncrossing partitions $\mathrm{NC}(n-k+1)$. Finally, we know that in each noncrossing partition of $S_{m,n}$, the point x_{m+1} must either be a singleton or share a block with x_m or one of $y_1, \ldots, y_n$, so $\mathrm{NC}(S_{m,n})$ is the union of $A, B_1, \ldots, B_n$ and it is clear from the definition that these subposets are disjoint. □

Our first application of Lemma 4.1 is to show that $\mathrm{NC}(S_{m,n})$ has a symmetric chain decomposition (and is therefore rank-symmetric).

Theorem 4.2. $\mathrm{NC}(S_{m,n})$ *admits a symmetric chain decomposition.*

Proof. As base cases, observe that $\mathrm{NC}(S_{0,n})$ is the classical lattice of noncrossing partitions $\mathrm{NC}(n)$ and $\mathrm{NC}(S_{m,0})$ is the Boolean lattice $\mathrm{BOOL}(m+1)$, each of which is known to admit a symmetric chain decomposition. Proceeding by induction on both m and n, suppose that $\mathrm{NC}(S_{a,b})$ has a symmetric chain decomposition when $a < m$ and $b \leq n$. When $m \geq 1$ and $n \geq 1$, Lemma 4.1 tells us that $\mathrm{NC}(S_{m,n})$ is the disjoint union of centered subposets which are isomorphic to $\mathrm{NC}(S_{m-1,n}) \times \mathrm{BOOL}(1)$ and $\mathrm{NC}(S_{m-1,k-1}) \times \mathrm{NC}(n-k+1)$. Using the inductive hypothesis and the facts that Boolean lattices and classical lattices of noncrossing partitions each have symmetric chain decompositions, we know by Remark 2.8 that $\mathrm{NC}(S_{m,n})$ has a symmetric chain decomposition as well. □

In Table 2, we provide the size of $\mathrm{NC}(S_{m,n})$ for small values of m and n. We are not aware of any previous appearances of these numbers. Finally, we establish a multivariate generating function which enumerates the noncrossing partitions of

$S_{m,n}$. As in the previous section, it would be interesting to find a closed enumeration formula.

Theorem 4.3. *If $s_{m,n}$ is the size of $\mathrm{NC}(S_{m,n})$, then the associated multivariate generating function is*

$$S(x,y) = \sum_{m,n\geq 0} s_{m,n}x^m y^n = \left(\frac{C(y)-1-y}{y^2}\right)\left(\frac{1}{1-x(1+C(y))}\right),$$

where $C(y) = (1-\sqrt{1-4y})/2y$ is the generating function for the Catalan numbers.

Proof. To start, we write

$$\begin{aligned}
S(x,y) &= \sum_{\substack{m\geq 0\\ n\geq 0}} s_{m,n}x^m y^n \\
&= s_{0,0} + \sum_{m\geq 1} s_{m,0}x^m + \sum_{n\geq 1} s_{0,n}y^n + \sum_{\substack{m\geq 1\\ n\geq 1}} s_{m,n}x^m y^n \\
&= 2 + \sum_{m\geq 1} 2^{m+1}x^m + \sum_{n\geq 1} C_{n+2}y^n + \sum_{\substack{m\geq 1\\ n\geq 1}} s_{m,n}x^m y^n \\
&= 2 + 4x\sum_{m\geq 0} 2^m x^m + \sum_{n\geq 3} C_n y^{n-2} + \sum_{\substack{m\geq 1\\ n\geq 1}} s_{m,n}x^m y^n \\
&= 2 + \frac{4x}{1-2x} + \frac{C(y)-1-y-2y^2}{y^2} + \sum_{\substack{m\geq 1\\ n\geq 1}} s_{m,n}x^m y^n \\
&= \frac{4x}{1-2x} + \frac{C(y)-1-y}{y^2} + \sum_{\substack{m\geq 1\\ n\geq 1}} s_{m,n}x^m y^n. \qquad (2)
\end{aligned}$$

By Lemma 4.1, we have the recursive formula

$$s_{m,n} = 2s_{m-1,n} + \sum_{k=1}^{n} s_{m-1,k-1}C_{n-k+1}$$

for $m,n \geq 1$, which can be rewritten as

$$s_{m,n} = s_{m-1,n} + \sum_{k=0}^{n} s_{m-1,k}C_{n-k}.$$

We can then use this formula to write

$$\begin{aligned}
\sum_{\substack{m\geq 1\\ n\geq 1}} s_{m,n}x^m y^n &= \sum_{\substack{m\geq 1\\ n\geq 1}} s_{m-1,n}x^m y^n + \sum_{\substack{m\geq 1\\ n\geq 1}}\sum_{k=0}^{n} s_{m-1,k}C_{n-k}x^m y^n \\
&= x\sum_{\substack{m\geq 0\\ n\geq 1}} s_{m,n}x^m y^n + x\sum_{\substack{m\geq 0\\ n\geq 1}}\sum_{k=0}^{n} s_{m,k}C_{n-k}x^m y^n.
\end{aligned}$$

The first term on the righthand side is

$$x \sum_{\substack{m\geq 0\\ n\geq 1}} s_{m,n} x^m y^n = x\left(S(x,y) - \sum_{m\geq 0} s_{m,0}x^m\right)$$
$$= x\left(S(x,y) - \frac{2}{1-2x}\right)$$

and the second term is

$$x \sum_{\substack{m\geq 0\\ n\geq 1}} \sum_{k=0}^{n} s_{m,k} C_{n-k} x^m y^n = x\left(\left(\sum_{\substack{m\geq 0\\ n\geq 0}} s_{m,n}x^m y^n\right)\left(\sum_{n\geq 0} C_n y^n\right) - \sum_{m\geq 0} s_{m,0}x^m\right)$$
$$= x\left(S(x,y)C(y) - \frac{2}{1-2x}\right).$$

Returning to Equation 2, we can plug in and cancel terms to obtain

$$S(x,y) = \frac{C(y)-1-y}{y^2} + xS(x,y) + xS(x,y)C(y)$$
$$S(x,y)(1-x(1+C(y)) = \frac{C(y)-1-y}{y^2}$$
$$S(x,y) = \left(\frac{C(y)-1-y}{y^2}\right)\left(\frac{1}{1-x(1+C(y))}\right),$$

which completes the proof. □

References

[1] B. Baumeister, K.-U. Bux, F. Götze, D. Kielak, and H. Krause. Non-crossing partitions. In *Spectral structures and topological methods in mathematics*, EMS Ser. Congr. Rep., pages 235–274. EMS Publ. House, Zürich, 2019.

[2] David Bessis. The dual braid monoid. *Annales Scientifiques de l'École Normale Supérieure*, 36(5):647–683, 2003.

[3] Thomas Brady and Colum Watt. $K(\pi, 1)$'s for Artin groups of finite type. *Geometriae Dedicata*, 94(1):225–250, 2002.

[4] Stella Cohen, Michael Dougherty, Andrew D. Harsh, and Spencer Park Martin. Noncrossing partition lattices from planar configurations. *Discrete Comput. Geom.*, 2024.

[5] N. G. de Bruijn, Ca. van Ebbenhorst Tengbergen, and D. Kruyswijk. On the set of divisors of a number. *Nieuw Arch. Wiskunde (2)*, 23:191–193, 1951.

[6] G. Kreweras. Sur les partitions non croisées d'un cycle. *Discrete Math.*, 1(4):333–350, 1972.

[7] Jon McCammond. Noncrossing partitions in surprising locations. *Amer. Math. Monthly*, 113(7):598–610, 2006.

[8] OEIS Foundation Inc. The On-Line Encyclopedia of Integer Sequences, 2025. Published electronically at `http://oeis.org`.

[9] Rodica Simion and Daniel Ullman. On the structure of the lattice of noncrossing partitions. *Discrete Math.*, 98(3):193–206, 1991.

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