CRITICAL POINTS, CRITICAL VALUES, AND A DETERMINANT IDENTITY FOR COMPLEX POLYNOMIALS

MICHAEL DOUGHERTY AND JON MCCAMMOND

Abstract. Given any n-tuple of complex numbers, one can easily define a canonical polynomial of degree $n+1$ that has the entries of this n-tuple as its critical points. In 2002, Beardon, Carne, and Ng studied a map $\theta\colon \mathbb{C}^n \to \mathbb{C}^n$ which outputs the critical values of the canonical polynomial constructed from the input, and they prove that this map is onto. Along the way, they show that θ is a local homeomorphism whenever the entries of the input are distinct and nonzero, and, implicitly, they produce a polynomial expression for the Jacobian determinant of θ. In this article we extend and generalize both the local homeomorphism result and the elegant determinant identity to analogous situations where the critical points occur with multiplicities. This involves stratifying $\mathbb{C}^n$ according to which coordinates are equal and generalizing θ to a similar map $\mathbb{C}^\ell \to \mathbb{C}^\ell$ where ℓ is the number of distinct critical points. The more complicated determinant identity that we establish is closely connected to the multinomial identity known as Dyson's conjecture.

1. Introduction

Let $p(z) \in \mathbb{C}[z]$ be a polynomial and let $z_0 \in \mathbb{C}$ be a complex number. If $p'(z_0) = 0$, then z_0 is a *critical point of* p and its image $p(z_0)$ is a *critical value of* p. For any n-tuple $\mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n$, one can define a canonical polynomial $p = p_{\mathbf{z}}$ that has the entries of $\mathbf{z}$ as its critical points:

$$p(z) = p_{\mathbf{z}}(z) = \int_0^z (w - z_1) \cdots (w - z_n)\ dw.$$

Using this polynomial, define the map $\theta\colon \mathbb{C}^n \to \mathbb{C}^n$ which sends $\mathbf{z} = (z_1, \ldots, z_n)$ to $\theta(\mathbf{z}) = (\theta_1(\mathbf{z}), \ldots, \theta_n(\mathbf{z})) = (p(z_1), \ldots, p(z_n))$ and note that θ sends the critical points of p to the critical values of p. In their 2002 article, Beardon, Carne, and Ng show that every n-tuple of complex numbers arises as the critical values of some polynomial by proving that this map θ is surjective [BCN02].

The Jacobian matrix $\mathbf{J} = \mathbf{J}(\mathbf{z})$ of the map θ at $\mathbf{z}$ is the $n \times n$ matrix with (i,j)-entry given by $(\mathbf{J})_{ij} = \frac{\partial}{\partial z_i}\theta_j(\mathbf{z}) = \frac{\partial}{\partial z_i} p(z_j)$. As part of their argument, Beardon, Carne, and Ng prove the following theorem regarding $\mathbf{J}$.

Theorem 1.1 ([BCN02]). *Let $\mathbf{z} \in \mathbb{C}^n$ and let $\mathbf{J} = \mathbf{J}(\mathbf{z})$ be the Jacobian matrix defined above. Then $\mathbf{J}$ is invertible if and only if $\mathbf{z}$ has distinct entries.*

Our first theorem provides an alternate proof via an explicit computation for the determinant of $\mathbf{J}$. For any positive integer n, we use $[n]$ to denote the set $\{1, \ldots, n\}$.

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Theorem A. *Let $\mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n$ and let $\mathbf{J} = \mathbf{J}(\mathbf{z})$ be the Jacobian matrix defined above. The Jacobian determinant factors as follows:*

$$\det \mathbf{J} = \frac{1}{n!} \left(\prod_{j \in [n]} (-z_j) \right) \left(\prod_{\substack{j,k \in [n] \\ j \neq k}} (z_k - z_j) \right).$$

Thus $\mathbf{J}$ is invertible if and only if $z_1, \ldots, z_n$ are distinct and nonzero.

We prove a generalization of this determinant identity by focusing on the case where the critical points have specified multiplicities. Throughout the article, we use the notation $\mathbf{a} = (a_1, \ldots, a_m)$ to denote an m-tuple of positive integers with sum $n = a_1 + \cdots + a_m$.

Definition 1.2. Let $\mathbf{z} = (z_1, \ldots, z_m) \in \mathbb{C}^m$ and define the polynomial $p_\mathbf{a} \in \mathbb{C}[z]$:

$$p_\mathbf{a}(z) = p_{\mathbf{z},\mathbf{a}}(z) = \int_0^z (w - z_1)^{a_1} \cdots (w - z_m)^{a_m} \, dw.$$

Then $p_\mathbf{a}$ is the unique polynomial of degree $n+1$ which has monic derivative, no constant term, and has z_i as a critical point with multiplicity a_i for each $i \in [m]$. The reader should note that one can view $z_1, \ldots, z_m$ as complex variables (rather than specific complex numbers) and then regard $p_\mathbf{a}$ as a polynomial in the ring $R[z]$, where $R = \mathbb{C}[z_1, \ldots, z_m]$. In the remainder of the article, we switch between these two viewpoints as needed. With this in mind, define $\theta_\mathbf{a} \colon \mathbb{C}^m \to \mathbb{C}^m$ to be the map which sends $(z_1, \ldots, z_m)$ to $(p_\mathbf{a}(z_1), \ldots, p_\mathbf{a}(z_m))$. Then the Jacobian matrix for $\theta_\mathbf{a}$ is the $m \times m$ matrix $\mathbf{J}_\mathbf{a}$ with (i,j)-entry $(\mathbf{J}_\mathbf{a})_{ij} = \frac{\partial}{\partial z_i} p_\mathbf{a}(z_j)$.

Our main theorem describes a factorization for the determinant of $\mathbf{J}_\mathbf{a}$.

Theorem B. *Let $\mathbf{z} = (z_1, \ldots, z_m) \in \mathbb{C}^m$ and let $\mathbf{J}_\mathbf{a} = \mathbf{J}_\mathbf{a}(\mathbf{z})$ be the Jacobian matrix defined above. The Jacobian determinant factors as follows:*

$$\det \mathbf{J}_\mathbf{a} = \frac{1}{\binom{n}{a_1, \ldots, a_m}} \left(\prod_{j \in [m]} (-z_j)^{a_j} \right) \left(\prod_{\substack{j,k \in [m] \\ j \neq k}} (z_k - z_j)^{a_j} \right).$$

Thus $\mathbf{J}_\mathbf{a}$ is invertible if and only if $z_1, \ldots, z_m$ are distinct and nonzero.

By the Inverse Function Theorem, the determinant result proven in [BCN02] implies that the map $\theta \colon \mathbb{C}^n \to \mathbb{C}^n$ is a local homeomorphism at points with distinct nonzero entries. Theorem B allows us to extend this local homeomorphism result to points with nondistinct but nonzero entries. To do so, we stratify the points in $\mathbb{C}^n$ with nonzero entries according to which entries are equal, where the strata are in bijection with the partitions of $[n]$. In this setting, Theorem B demonstrates that while θ may or may not be a local homeomorphism at a generic point with nonzero entries, the restriction of θ to the corresponding stratum *is* a local homeomorphism.

Theorem C. *Let λ be a partition of $[n]$ and let $\mathbb{C}^\lambda$ be the subspace of $\mathbb{C}^n$ consisting of all points $\mathbf{z} = (z_1, \ldots, z_n)$ where $z_i = z_j$ if and only if i and j belong to the same block. Then θ restricts to a map $\theta_\lambda \colon \mathbb{C}^\lambda \to \overline{\mathbb{C}^\lambda}$, where $\overline{\mathbb{C}^\lambda}$ is the closure of $\mathbb{C}^\lambda$, and θ_λ is a local homeomorphism at $\mathbf{z} \in \mathbb{C}^\lambda$ if the entries of $\mathbf{z}$ are nonzero.*

These theorems have direct connections to the braid group as the fundamental group of the space of complex polynomials with distinct roots. The details will be given in upcoming work by the authors. The article is organized into five sections. Section 2 provides integration formulas for products of polynomials. Section 3 discusses monomial orders for multivariable polynomials and the well-known Vandermonde determinant, which serves as a guide for our proofs of Theorems A and B in Section 4. We then describe a connection between our determinant result and the multinomial identity known as Dyson's conjecture in Section 5 before finally proving Theorem C in Section 6.

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2. Integrating Products

We begin with a simple way to antidifferentiate products of polynomials.

Definition 2.1 (Derivative sequences)**.** Let R be a commutative ring. A sequence of polynomials $(f_*) = (f_0, f_1, f_2, \ldots)$ in $R[z]$ is a *derivative sequence* if $f_n(z)$ has degree n and $\frac{d}{dz} f_n(z) = f_{n-1}(z)$ for each positive integer n. For convenience, we define $f_n(z)$ to be the zero polynomial if n is a negative integer.

Derivative sequences are similar to *Appell sequences*, which instead require the condition $\frac{d}{dz} f_n(z) = n \cdot f_{n-1}(z)$. The tools presented in this section appear similar to others in this related context; see [Lee11] and [LPZ14] for examples.

Example 2.2 (A special derivative sequence)**.** If we fix some $z_0 \in \mathbb{C}$ and define $f_n(z) = \frac{1}{n!}(z - z_0)^n$, then (f_*) is a derivative sequence in $\mathbb{C}[z]$ with the unusual property that $f_n(z)$ is a factor of $f_m(z)$ for all $m \geq n$.

Every polynomial belongs to a derivative sequence, and the product of two polynomials has an antiderivative which is easily expressed using the derivative sequences to which they belong.

Example 2.3 (Derivative sequences and antiderivatives)**.** Let f_5 and g_3 be polynomials of degree 5 and 3 respectively and let (f_*) and (g_*) be derivative sequences that contain them. We can use these derivative sequences to produce two antiderivatives for the product $f_5 \cdot g_3$. Consider $(f_6 \cdot g_3) - (f_7 \cdot g_2) + (f_8 \cdot g_1) - (f_9 \cdot g_0)$. To see this is an antiderivative, simply expand $\frac{d}{dz}[(f_6 \cdot g_3) - (f_7 \cdot g_2) + (f_8 \cdot g_1) - (f_9 \cdot g_0)]$ to $(f_5 \cdot g_3 + f_6 \cdot g_2) - (f_6 \cdot g_2 + f_7 \cdot g_1) + (f_7 \cdot g_1 + f_8 \cdot g_0) - (f_8 \cdot g_0 + f_9 \cdot g_{-1})$ which simplifies to $f_5 \cdot g_3 - f_9 \cdot g_{-1} = f_5 \cdot g_3$ since g_{-1} is the zero polynomial. Switching the role of f and g, we see that $(f_5 \cdot g_4) - (f_4 \cdot g_5) + (f_3 \cdot g_6) - (f_2 \cdot g_7) + (f_1 \cdot g_8) - (f_0 \cdot g_9)$ is another antiderivative of $f_5 \cdot g_3$.

The antiderivatives of a product listed in Example 2.3 are just the result of iterated integration by parts. The derivative sequences merely predetermine the antiderivatives that one uses.

Lemma 2.4 (An antiderivative formula)**.** *If (f_*) and (g_*) are derivative sequences, then $r(z) = \sum_{i=0}^{b} (-1)^i (f_{a+1+i}(z) \cdot g_{b-i}(z))$ is an antiderivatve of $f_a(z) \cdot g_b(z)$. Note that if (f_*) is the derivative sequence with $f_n(z) = \frac{1}{n!}(z - z_0)^n$, then $(z - z_0)^{a+1}$ is a factor of the polynomial $r(z)$.*

Proof. Computing the derivative is straightforward. We apply the product rule, split into two summands, and reindex; the result is that all terms cancel except the first, which is what we want, and the last, which is zero. In symbols:

$$\frac{d}{dz}\left(\sum_{i=0}^{b}(-1)^i(f_{a+1+i}\cdot g_{b-i})\right)=\sum_{i=0}^{b}(-1)^i(f_{a+i}\cdot g_{b-i}+f_{a+1+i}\cdot g_{b-i-1})$$

$$\begin{aligned}
&=\left(\sum_{i=0}^{b}(-1)^i(f_{a+i}\cdot g_{b-i})\right)+\left(\sum_{i=0}^{b}(-1)^i(f_{a+i+1}\cdot g_{b-i-1})\right)\\
&=\left(\sum_{i=0}^{b}(-1)^i(f_{a+i}\cdot g_{b-i})\right)-\left(\sum_{i=1}^{b+1}(-1)^i(f_{a+i}\cdot g_{b-i})\right)\\
&=f_a\cdot g_b-f_{a+b+1}\cdot g_{-1}\\
&=f_a\cdot g_b.
\end{aligned}$$

The final assertion follows from the fact that $(z-z_0)^{a+1}$ is a factor of $f_j(z)$ for all $j>a$ in this special case. ☐

For later use we include a specific application of Lemma 2.4.

Proposition 2.5. *Let z_0 be a complex variable and let a and b be positive integers with $a+b=n$. If $p(z)=\int_0^z w^a(w-z_0)^b dw$, then $p(z_0)=(-1)^b\frac{a!\,b!}{(n+1)!}z_0^{n+1}$ and $\frac{\partial}{\partial z_0}(p(z_0))=(-1)^b\frac{a!\,b!}{n!}z_0^n$.*

Proof. For any nonnegative integers i and j, define the polynomials $f_i(z)=\frac{1}{i!}z^i$ and $g_j(z)=\frac{1}{j!}(z-z_0)^j$. Then (f_*) and (g_*) are derivative sequences and by Lemma 2.4, we know that $f_{a+1}\cdot g_b-f_{a+2}\cdot g_{b-1}+\cdots+(-1)^b f_{n+1}\cdot g_0$ is an antiderivative for the product $f_a\cdot g_b$. Notice that $p(z)=a!\,b!\int_0^z(f_a(w)\cdot g_b(w))\,dw$ and $f_i(0)=0$ for all $i>0$, so we have that

$$p(z)=a!\,b!\big(f_{a+1}(z)\cdot g_b(z)-f_{a+2}(z)\cdot g_{b-1}(z)+\cdots+(-1)^b f_{n+1}(z)\cdot g_0(z)\big).$$

Finally, we note that $g_j(z_0)=0$ for all $j>0$ while $g_0(z_0)=1$, so

$$p(z_0)=a!\,b!(-1)^b(f_{n+1}(z_0)\cdot g_0(z_0))=(-1)^b\frac{a!\,b!}{(n+1)!}z_0^{n+1}$$

as desired. ☐

The following example outlines our primary motivation for this section.

Example 2.6 (Emphasizing a factor)**.** Let R be the polynomial ring $\mathbb{C}[z_1,\ldots,z_m]$ and define the polynomial $\tilde{p}_{\mathbf{a}}(z)\in R[z]$ by

$$\tilde{p}_{\mathbf{a}}(z)=\int_0^z\frac{(w-z_1)^{a_1}}{a_1!}\cdots\frac{(w-z_m)^{a_m}}{a_m!}\,dw.$$

Notice that $(a_1!\cdots a_m!)\tilde{p}_{\mathbf{a}}(z)$ is equal to the polynomial $p_{\mathbf{a}}(z)$ defined in the introduction. For any choice of $j\in[m]$, define two derivative sequences (f_*) and (g_*); define (f_*) explicitly by $f_k(w)=\frac{1}{k!}(w-z_j)^k$ for each nonnegative integer k, and let (g_*) be any derivative sequence containing the term

$$g_{n-a_j}(w)=\frac{(w-z_1)^{a_1}}{a_1!}\cdots\frac{(w-z_{j-1})^{a_{j-1}}}{a_{j-1}!}\frac{(w-z_{j+1})^{a_{j+1}}}{a_{j+1}!}\cdots\frac{(w-z_m)^{a_m}}{a_m!}.$$

We write $\tilde{p}_{\mathbf{a},j}$ to denote the antiderivative of the product $f_{a_j} \cdot g_{n-a_j}$ given by Lemma 2.4. Then $\tilde{p}_{\mathbf{a}}(z) = \tilde{p}_{\mathbf{a},j}(z) - \tilde{p}_{\mathbf{a},j}(0)$ and we say that the index j has been "emphasized". We make several observations about these expressions:

(1) $\tilde{p}_{\mathbf{a},j}(z_j) = 0$
(2) $\tilde{p}_{\mathbf{a},j}(0)$ is divisible by $z_j^{a_j+1}$
(3) $\tilde{p}_{\mathbf{a},j}(z_i)$ is divisible by $(z_i - z_j)^{a_i+a_j+1}$

To see this, first recall by Lemma 2.4 that $\tilde{p}_{\mathbf{a},j}(w)$ is divisible by $(w-z_j)^{a_j+1}$. From here, we can substitute $w = z_j$ or $w = 0$ to prove (1) or (2), respectively. As for (3), notice that since $(w - z_i)^{a_i}$ is a factor of g_{n-a_j}, we know that $g_{n-a_j-\ell}(z_i) = 0$ for all $\ell < a_i$. Therefore, the nonzero terms of $\tilde{p}_{\mathbf{a},j}(z_i)$ are those of the form $(-1)^\ell (f_{a_j+\ell+1}(z_i) \cdot g_{n-a_j-\ell}(z_i))$ where $\ell \geq a_i$, and in this setting, each $f_{a_j+\ell+1}(z)$ is divisible by $(z_i - z_j)^{a_i+a_j+1}$, so we are done.

3. Ordering Monomials and the Vandermonde Determinant

In this section we review how one computes the Vandermonde determinant as a warmup to the proofs of Theorems A and B in Section 4. Our proof uses the idea of a monomial order borrowed from the standard construction of a Gröbner basis.

Definition 3.1 (Monomial order)**.** Fix a positive integer m and a polynomial ring $R = \mathbb{C}[z_1, \ldots, z_m]$. Ordering the variables $z_1 < z_2 < \cdots < z_m$ then yields a lexicographic order on the monomials of R. One first compares the degree of z_m in the two terms and orders them accordingly. If they agree, one compares the degrees of z_{m-1} and so on. For any polynomial $f \in R$, the *leading term* $\textsc{lt}(f)$ is the summand of f containing the largest monomial in this order and the coefficient of the leading term is the *leading coefficient*.

Example 3.2. Let $R = \mathbb{C}[z_1, z_2, z_3]$ and define $f \in R$ by $f = 7z_1^9 z_2^6 z_3^3 - 6z_1 z_2^2 z_3^4$. Then $z_1^9 z_2^6 z_3^3 < z_1 z_2^2 z_3^4$ in the lexicographic order on R, so $\textsc{lt}(f) = -6z_1 z_2^2 z_3^4$ and the leading coefficient is -6.

The notion of a leading term is useful in the evaluation of the Vandermonde determinant, traditionally introduced to show that any $n+1$ distinct points on the rational normal curve are in general position. Recall that if $\mathbf{M}$ is an $n \times n$ matrix, then the *Leibniz determinant formula* computes the determinant of $\mathbf{M}$ as a sum over permutations σ in the symmetric group $\textsc{Sym}_n$:

$$\det \mathbf{M} = \sum_{\sigma \in \textsc{Sym}_n} \operatorname{sgn}(\sigma) \prod_{i \in [n]} (\mathbf{M})_{\sigma(i),i}.$$

While we do not apply the following theorem in this article, its proof is a valuable illustration of techniques we employ when proving Theorem B.

Theorem 3.3 (Vandermonde determinant)**.** *If* $\mathbf{V}$ *is the Vandermonde matrix*

$$\mathbf{V} = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ z_1 & z_2 & z_3 & \cdots & z_{n-1} & z_n \\ z_1^2 & z_2^2 & z_3^2 & \cdots & z_{n-1}^2 & z_n^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ z_1^{n-2} & z_2^{n-2} & z_3^{n-2} & \cdots & z_{n-1}^{n-2} & z_n^{n-2} \\ z_1^{n-1} & z_2^{n-1} & z_3^{n-1} & \cdots & z_{n-1}^{n-1} & z_n^{n-1} \end{pmatrix}$$

with (i,j)-entry z_j^{i-1}, then its determinant is

$$\det \mathbf{V} = \prod_{\substack{j,k\in[n] \\ j<k}} (z_k - z_j).$$

Proof. We view the entries as monomials in the variables $z_1, \ldots, z_n$ and we let $D(\mathbf{z}) \in \mathbb{C}[z_1, \ldots, z_n]$ denote the multivariable polynomial on the righthand side of the claimed formula. Since the entries in the i-th row of $\mathbf{V}$ are homogeneous polynomials of degree $i-1$, we know that each summand in the Leibniz formula for $\det \mathbf{V}$ is homogeneous of degree $0+1+\cdots+(n-1)$ and thus $\det \mathbf{V}$ is a homogeneous polynomial of degree at most $\binom{n}{2}$. Moreover, the determinant is unchanged if the j-th column is subtracted from the k-th column, and since every entry in the new k-th column is divisible by $z_k - z_j$, this expression divides the determinant. Since the linear factors that arise as j and k are varied represent distinct non-associate prime elements of the polynomial ring $\mathbb{C}[z_1, \ldots, z_n]$, we know that their product, $D(\mathbf{z})$, divides $\det \mathbf{V}$ as well. Because $D(\mathbf{z})$ is also homogeneous with degree $\binom{n}{2}$, we know that for some constant C, $\det \mathbf{V}$ is of the form $\det \mathbf{V} = C \cdot D(\mathbf{z})$.

To determine the value of C, we lexicographically order the monomials as in Definition 3.1 and then compare the leading terms of $\det \mathbf{V}$ and $D(\mathbf{z})$. In the latter case, the leading term is $\prod_{k\in[n]} z_k^{k-1}$, obtained by always choosing z_k instead of z_j when expanding the product. Similarly, the leading term of $\det \mathbf{V}$ appears in only one summand of the Leibniz formula, corresponding to when σ is the identity permutation. To see this, first notice that $(\mathbf{V})_{nn}$ is the only entry which contributes a large enough power of z_n to the determinant, so we must have $\sigma(n) = n$. We then induct by observing that the $(n-1) \times (n-1)$ submatrix of $\mathbf{V}$ obtained by deleting the last row and column is simply a smaller Vandermonde matrix. Therefore, $\textsc{lt}(\det \mathbf{V})$ is also $\prod_{k\in[n]} z_k^{k-1}$, the product of the diagonal entries. So $C = 1$ and we are done. □

In more complicated cases, we can use the Leibniz formula to compute the leading term of a determinant.

Proposition 3.4 (Leading terms of determinants)**.** *Let $R = \mathbb{C}[z_1, \ldots, z_m]$ be lexicographically ordered and let $\mathbf{M}$ be a $k \times k$ matrix with entries in R. If $\mathbf{M}'$ is the $k \times k$ matrix with (i,j)-entry $(\mathbf{M}')_{ij} = \textsc{lt}((\mathbf{M})_{ij})$, then $\textsc{lt}(\det \mathbf{M}) = \textsc{lt}(\det \mathbf{M}')$.*

Proof. The proof is immediate from the Leibniz formula and the fact that the leading term of a product is the product of the leading terms. □

4. A Determinant Identity

In this section, we prove Theorems A and B, along with several technical propositions and lemmas. Note that Theorem A follows immediately from Theorem B by choosing $\mathbf{a}$ to be the n-tuple $(1, \ldots, 1)$, so it suffices to prove Theorem B. We begin by illustrating the proof technique with an example, then outline the proof before finally providing the supporting technical lemmas and propositions.

Example 4.1 (Two variables with multiplicity)**.** It is easy to compute the determinant in small cases using a software package such as *SageMath*. Consider, for example, the case where $n = 2$ and $\mathbf{a} = (2, 3)$, and for readability we write $\mathbf{z} = (x, y)$

instead of $\mathbf{z} = (z_1, z_2)$. First we compute the polynomial $p_{\mathbf{a}}(z)$, with the details omitted:

$$\begin{aligned} p_{\mathbf{a}}(z) &= \int_0^z (w-x)^2(w-y)^3 \; dw \\ &= \frac{z^6}{6} - (2x+3y)\frac{z^5}{5} + (x^2+6xy+3y^2)\frac{z^4}{4} \\ &\quad - (3x^2y+6xy^2+y^3)\frac{z^3}{3} + (3x^2y^2+2xy^3)\frac{z^2}{2} - (x^2y^3)z. \end{aligned}$$

Next we compute the map $\theta_{\mathbf{a}} \colon \mathbb{C}^2 \to \mathbb{C}^2$:

$$\begin{aligned} \theta_{\mathbf{a}}(x,y) &= (p_{\mathbf{a}}(x), p_{\mathbf{a}}(y)) \\ &= \left(\frac{1}{60}x^6 - \frac{1}{10}x^5y + \frac{1}{4}x^4y^2 - \frac{1}{3}x^3y^3, -\frac{1}{4}x^2y^4 + \frac{1}{10}xy^5 - \frac{1}{60}y^6\right). \end{aligned}$$

Then we compute the Jacobian $\mathbf{J}_{\mathbf{a}}$:

$$\mathbf{J}_{\mathbf{a}} = \begin{pmatrix} \frac{1}{10}\,x^5 - \frac{1}{2}\,x^4y + x^3y^2 - x^2y^3 & -\frac{1}{2}\,xy^4 + \frac{1}{10}\,y^5 \\ -\frac{1}{10}\,x^5 + \frac{1}{2}\,x^4y - x^3y^2 & -x^2y^3 + \frac{1}{2}\,xy^4 - \frac{1}{10}\,y^5 \end{pmatrix}.$$

And finally we compute the determinant of $\mathbf{J}_{\mathbf{a}}$:

$$\det \mathbf{J}_{\mathbf{a}} = -\frac{1}{10}x^2y^3(x-y)^5.$$

One could instead compute this determinant from three observations. First, notice that the first column is divisible by x^2 and the second column by y^3. Next, if we subtract the second column from the first and factor, then each resulting entry is divisible by $(x-y)^5$. Thus $\det \mathbf{J}_{\mathbf{a}}$ is a homogeneous polynomial of degree 10 which is divisible by $x^2y^3(x-y)^5$, so

$$\det \mathbf{J}_{\mathbf{a}} = Cx^2y^3(x-y)^5$$

for some constant C. By inspection, the leading term of $x^2y^3(x-y)^5$ is $-x^2y^8$ while the leading term of $\det \mathbf{J}_{\mathbf{a}}$ is $\frac{1}{10}x^2y^8$. Therefore, $C = \frac{-1}{10}$ and we are done.

To emphasize its straightforward structure, we give the proof of Theorem B here before continuing on to prove the prerequisite lemmas.

Proof of Theorem B. The proof is in three steps. First, Proposition 4.2 tells us that $\det \mathbf{J}_{\mathbf{a}}$ is divisible by $z_j^{a_j}$ for each $j \in [m]$. Second, Proposition 4.3 implies that the determinant is divisible by $(z_k - z_j)^{a_j + a_k}$ for each $j < k$ in $[m]$. Since these polynomials are powers of distinct non-associate prime elements in the polynomial ring $\mathbb{C}[z_1, \dots, z_m]$ and we know that $\det \mathbf{J}_{\mathbf{a}}$ is a homogeneous polynomial of degree at most $n(a_1 + \cdots + a_m)$, it must be that $\det \mathbf{J}_{\mathbf{a}}$ is a constant multiple of their product. After rewriting $(z_k - z_j)^{a_j+a_k} = (-1)^{a_k}(z_k - z_j)^{a_j}(z_j - z_k)^{a_k}$ and absorbing powers of -1 into the constant, we may write $\det \mathbf{J}_{\mathbf{a}} = C \cdot D(\mathbf{z})$, where C is a constant and

$$D(\mathbf{z}) = \left(\prod_{j \in [m]} (-z_j)^{a_j}\right)\left(\prod_{\substack{j,k \in [m] \\ j \neq k}} (z_k - z_j)^{a_j}\right).$$

Lastly, we determine the value of C by comparing the leading coefficients of $\det \mathbf{J}_{\mathbf{a}}$ and $D(\mathbf{z})$. We can see that $\textsc{lt}(D(\mathbf{z}))$ has coefficient $(-1)^{a_1+2a_2+\cdots+ma_m}$, obtained by always choosing the higher-index terms when expanding the product $(z_k - z_j)^{a_j}$. On the other hand, we know by Proposition 4.7 that $\textsc{lt}(\det \mathbf{J_a})$ has coefficient $(-1)^{a_1+2a_2+\cdots+ma_m}\binom{n}{a_1,\ldots,a_m}^{-1}$. Thus, $C = \binom{n}{a_1,\ldots,a_m}^{-1}$ and we are done. □

We now prove several lemmas and the three propositions used in the proof of Theorem B. The first two propositions are straightforward, while the third is more complicated.

To start, we observe that by Definition 1.2 and Example 2.6, the (i,j)-entry of the Jacobian matrix $J_{\mathbf{a}}$ can be written as

$$(*) \qquad (\mathbf{J_a})_{ij} = \frac{\partial}{\partial z_i} p_{\mathbf{a}}(z_j) = (a_1! \cdots a_m!) \frac{\partial}{\partial z_i}\Big[\tilde{p}_{\mathbf{a},k}(z_j) - \tilde{p}_{\mathbf{a},k}(0)\Big].$$

Note that the freedom of choosing k allows this term to be expressed in many different ways without altering its value.

Proposition 4.2 (Columns)**.** *Each entry of the j-th column of $\mathbf{J_a}$ is divisible by $z_j^{a_j}$.*

Proof. Let $i, j \in [m]$. By Equation $(*)$, emphasizing z_j, we have the following:

$$(\mathbf{J_a})_{ij} = (a_1! \cdots a_m!) \frac{\partial}{\partial z_i}\Big[\tilde{p}_{\mathbf{a},j}(z_j) - \tilde{p}_{\mathbf{a},j}(0)\Big].$$

By the observations made in Example 2.6, we know that $\tilde{p}_{\mathbf{a},j}(z_j) = 0$ and $\tilde{p}_{\mathbf{a},j}(0)$ is divisible by $z_j^{a_j+1}$. Thus, $(\mathbf{J_a})_{ij}$ is divisible by $z_j^{a_j}$ if $i = j$ and $z_j^{a_j+1}$ otherwise. □

Proposition 4.3 (Column differences)**.** *If the j-th column of $\mathbf{J_a}$ is subtracted from the k-th column, then each entry of the new k-th column is divisible by $(z_k - z_j)^{a_k+a_j}$.*

Proof. For $i, j, k \in [m]$ we show that $(\mathbf{J_a})_{ik} - (\mathbf{J_a})_{ij}$ is divisible by $(z_k - z_j)^{a_k+a_j}$. Using Equation $(*)$, emphasizing z_j in both applications, we have the following:

$$\frac{(\mathbf{J_a})_{ik} - (\mathbf{J_a})_{ij}}{a_1! \cdots a_m!} = \frac{\partial}{\partial z_i}\Big[\tilde{p}_{\mathbf{a},j}(z_k) - \tilde{p}_{\mathbf{a},j}(0)\Big] - \frac{\partial}{\partial z_i}\Big[\tilde{p}_{\mathbf{a},j}(z_j) - \tilde{p}_{\mathbf{a},j}(0)\Big].$$

By canceling terms, we have that

$$(\mathbf{J_a})_{ik} - (\mathbf{J_a})_{ij} = (a_1! \cdots a_m!) \frac{\partial}{\partial z_i}\Big[\tilde{p}_{\mathbf{a},j}(z_k) - \tilde{p}_{\mathbf{a},j}(z_j)\Big],$$

and by Example 2.6, we know that $\tilde{p}_{\mathbf{a},j}(z_k)$ is divisible by $(z_k - z_j)^{a_k+a_j+1}$ and $\tilde{p}_{\mathbf{a},j}(z_j) = 0$. Therefore, $(\mathbf{J_a})_{ik} - (\mathbf{J_a})_{ij}$ is divisible by $(z_k - z_j)^{a_k+a_j}$ if $i \in \{j, k\}$ and $(z_k - z_j)^{a_k+a_j+1}$ otherwise, so we are done. □

The last proposition used in the proof of Theorem B concerns the coefficient of the leading term in $\det \mathbf{J_a}$. As a first step, we consider the highest power of z_m which appears in each entry of $\mathbf{J_a}$ and apply this to the leading term of $\det \mathbf{J_a}$.

Lemma 4.4 (Exponents of z_m)**.** *The highest exponent of z_m appearing in the entry $(\mathbf{J_a})_{ij}$ is a_m if $i, j < m$, $a_m - 1$ if $j < i = m$, and n if $j = m$.*

Proof. By Definition 1.2, we have the following:

$$\begin{aligned}(\mathbf{J_a})_{ij} &= \frac{\partial}{\partial z_i} p_{\mathbf{a}}(z_j) \\ &= \frac{\partial}{\partial z_i} \int_0^{z_j} (w - z_1)^{a_1} \cdots (w - z_m)^{a_m}\, dw.\end{aligned}$$

We now expand inside the integral to find the largest power of z_m. When $i = m$, the largest power appears in the term $\frac{\partial}{\partial z_m} \int_0^{z_j} w^{n-a_m}(-z_m)^{a_m}\ dw$, and thus the maximum exponent is $a_m - 1$ if $j < m$ and n if $j = m$. Similarly, when $i < m$, the largest power of z_m appears in the term $\frac{\partial}{\partial z_i} \int_0^{z_j} w^{n-a_m-1}(-z_i)(-z_m)^{a_m}\ dw$, and so the maximum exponent is a_m if $j < m$ and n if $j = m$. □

Example 4.5. Let $\mathbf{a} = (3, 7, 2, 6)$. Then the highest exponent of z_4 in the entry $(\mathbf{J_a})_{ij}$ is 6 if $i, j < 4$, 5 if $j < i = 4$, and 18 if $j = 4$. This information is easily visualized in the matrix

$$\begin{pmatrix} 6 & 6 & 6 & 18 \\ 6 & 6 & 6 & 18 \\ 6 & 6 & 6 & 18 \\ 5 & 5 & 5 & 18 \end{pmatrix}$$

where each entry contains the largest exponent of z_4 in the corresponding entry of $\mathbf{J_a}$. As a consequence, it is easy to see that the appearance of $\text{LT}(\det \mathbf{J_a})$ in the Leibniz formula is restricted to the summands which include the entry $(\mathbf{J_a})_{44}$.

Lemma 4.6 (Leading term of $\det \mathbf{J_a}$)**.** *The leading term of* $\det \mathbf{J_a}$ *may be computed recursively via the formula*

$$\text{LT}(\det \mathbf{J_a}) = (-z_m)^{a_m(m-1)} \cdot \text{LT}(\det \mathbf{J_{a'}}) \cdot \text{LT}((\mathbf{J_a})_{mm})$$

where $\mathbf{a}' = (a_1, \ldots, a_{m-1})$.

Proof. By Proposition 3.4, the leading term of $\det \mathbf{J_a}$ is equal to the leading term of the determinant of the matrix obtained by taking the leading term of each entry in $\mathbf{J_a}$. Since z_m is the highest-ordered variable, we know that $\text{LT}((\mathbf{J_a})_{ij})$ has the highest power of z_m in $(\mathbf{J_a})_{ij}$ as a factor. Thus, Lemma 4.4 and the Leibniz formula tell us that $\text{LT}(\det \mathbf{J_a})$ has a factor of $z_m^{a_m(m-1)+n}$, and this term is obtained from the Leibniz formula only in terms which include $(\mathbf{J_a})_{mm}$. As for the entries with $i, j < m$,

$$\begin{aligned}
\text{LT}((\mathbf{J_a})_{ij}) &= \text{LT}\left(\frac{\partial}{\partial z_i} \int_0^{z_j} (w - z_1)^{a_1} \cdots (w - z_m)^{a_m}\ dw\right) \\
&= \text{LT}\left(\frac{\partial}{\partial z_i} \int_0^{z_j} (w - z_1)^{a_1} \cdots (w - z_{m-1})^{a_{m-1}}(-z_m)^{a_m}\ dw\right) \\
&= \text{LT}\left((-z_m)^{a_m} \frac{\partial}{\partial z_i} \int_0^{z_j} (w - z_1)^{a_1} \cdots (w - z_{m-1})^{a_{m-1}}\ dw\right) \\
&= (-z_m)^{a_m} \text{LT}((\mathbf{J_{a'}})_{ij})
\end{aligned}$$

and so by the Leibniz formula, the claim is proven. □

We are now ready to compute the coefficient of the leading term of $\det \mathbf{J_a}$.

Proposition 4.7 (Constant coefficient)**.** *The leading term* $\text{LT}(\det \mathbf{J_a})$ *has constant coefficient* $(-1)^{a_1+2a_2+\cdots+ma_m}\binom{n}{a_1,\ldots,a_m}^{-1}$.

Proof. We prove by induction on m. When $m = 1$, we have $\mathbf{a} = (a_1)$ and

$$p_{\mathbf{a}}(z) = \int_0^z (w - z_1)^{a_1}\ dw = \frac{1}{a_1 + 1}\Big[(z - z_1)^{a_1+1} - (-z_1)^{a_1+1}\Big],$$

so $\mathbf{J_a}$ is a 1×1 matrix with a single entry:

$$(\mathbf{J_a})_{1,1} = \frac{d}{dz_1}\Big[p_{\mathbf{a}}(z_1)\Big] = \frac{d}{dz_1}\left[\frac{(-1)^{a_1} z_1^{a_1+1}}{a_1+1}\right] = (-1)^{a_1} z_1^{a_1}.$$

Thus $\text{LT}(\det \mathbf{J_a}) = (-1)^{a_1} z_1^{a_1}$ and so the coefficient is $(-1)^{a_1}$.

Now, let $\mathbf{a} = (a_1, \ldots, a_m)$ and suppose the claim holds for the $(m-1) \times (m-1)$ matrix $\mathbf{J_{a'}}$, where $\mathbf{a'} = (a_1, \ldots, a_{m-1})$. Then Lemma 4.6 tells us that

$$\text{LT}(\det \mathbf{J_a}) = (-z_m)^{a_m(m-1)} \cdot \text{LT}(\det \mathbf{J_{a'}}) \cdot \text{LT}((\mathbf{J_a})_{mm})$$

and by Proposition 2.5, we have the following:

$$\begin{aligned} \text{LT}((\mathbf{J_a})_{mm}) &= \text{LT}\left(\frac{\partial}{\partial z_m}\int_0^{z_m} (w-z_1)^{a_1}\cdots(w-z_m)^{a_m}\; dw\right) \\ &= \text{LT}\left(\frac{\partial}{\partial z_m}\int_0^{z_m} w^{n-a_m}(w-z_m)^{a_m}\; dw\right) \\ &= (-1)^{a_m}\frac{a_m!\,(n-a_m)!}{n!} z_m^n \end{aligned}$$

So the leading coefficient of $(\mathbf{J_a})_{mm}$ is

$$(-1)^{a_m}\frac{a_m!\,(n-a_m)!}{n!}$$

and by the inductive hypothesis, the leading coefficient of $\det \mathbf{J_{a'}}$ is

$$\left((-1)^{a_1+2a_2+\cdots+(m-1)a_{m-1}}\binom{a_1+\cdots+a_{m-1}}{a_1,\ldots,a_{m-1}}^{-1}\right).$$

Putting it all together, we have that the leading coefficient of $\det \mathbf{J_a}$ is

$$\begin{aligned} &(-1)^{a_1+2a_2+\cdots+ma_m}\frac{a_1!\cdots a_{m-1}!}{(n-a_m)!}\frac{a_m!(n-a_m)!}{n!} \\ &= (-1)^{a_1+2a_2+\cdots+ma_m}\binom{n}{a_1,\ldots,a_m}^{-1} \end{aligned}$$

as desired. □

5. Connections to Dyson's Conjecture

The polynomial expression presented in Theorem B is connected to a well-known multinomial identity, originally conjectured by the physicist Freeman Dyson [Dys62] and proven independently by Gunson [Gun62] and Wilson [Wil62]. A particularly short proof was later given by Good [Goo70]. We include its statement here to illustrate a connection between Theorem B and the existing literature.

Theorem 5.1 (Dyson's Conjecture)**.** *The constant term of the Laurent polynomial*

$$\prod_{\substack{j,k\in[m] \\ j\neq k}}\left(1-\frac{z_j}{z_k}\right)^{a_j}$$

is equal to the multinomial coefficient $\binom{n}{a_1,\ldots,a_m}$.

To see the connection with Theorem B, consider:

$$\begin{aligned}
\det \mathbf{J_a} &= \frac{1}{\binom{n}{a_1,\dots,a_m}} \left(\prod_{j\in[m]} (-z_j)^{a_j} \right) \left(\prod_{\substack{j,k\in[m] \\ j\neq k}} (z_k - z_j)^{a_j} \right) \\
&= \frac{1}{\binom{n}{a_1,\dots,a_m}} \left(\prod_{j\in[m]} (-z_j)^{a_j} \right) \left(\prod_{\substack{j,k\in[m] \\ j\neq k}} \left(z_k \left(1 - \frac{z_j}{z_k} \right) \right)^{a_j} \right) \\
&= \frac{(-1)^n}{\binom{n}{a_1,\dots,a_m}} \left(\prod_{j\in[m]} (z_j)^n \right) \left(\prod_{\substack{j,k\in[m] \\ i\neq j}} \left(1 - \frac{z_j}{z_k} \right)^{a_j} \right).
\end{aligned}$$

From here, we can see that Dyson's conjecture is equivalent to the fact that the monomial $\prod_{j\in[m]} z_j^n$ appears in $\det \mathbf{J_a}$ with coefficient $(-1)^n$. In other words, if we divide the j-th column of $\mathbf{J_a}$ by the monomial z_j^n for each j, then the determinant becomes a Laurent polynomial with constant term $(-1)^n$.

6. A Stratification for $\mathbb{C}^n$

As discussed in Section 4, the map $\theta\colon \mathbb{C}^n \to \mathbb{C}^n$ which sends each $\mathbf{z} = (z_1, \dots, z_n)$ to $(p(z_1), \dots, p(z_n))$ has $\mathbf{J}$ as its Jacobian matrix. By Theorem 1.1 and Theorem A, $\mathbf{J}$ is invertible if and only if the entries of $\mathbf{z}$ are distinct and nonzero. Together with the Inverse Function Theorem (see [GR65], for example), this implies that θ is a local homeomorphism at $\mathbf{z}$ if $z_1, \dots, z_n$ are distinct and nonzero. In this section, we provide a similar interpretation for Theorem B.

Definition 6.1 (Partitions)**.** Recall that if λ is a collection of ℓ nonempty pairwise disjoint sets with union equal to a given set, then λ is a *partition* of that set with ℓ *blocks*. The partition λ is a *refinement* of the partition μ (or *finer* than μ) if each block in μ is a union of blocks in λ.

Definition 6.2 (Stratifying $\mathbb{C}^n$)**.** For each $\mathbf{z} = (z_1, \dots, z_n) \in \mathbb{C}^n$, define $\textsc{Part}(\mathbf{z})$ to be the unique partition of $[n]$ such that i and j belong to the same block of λ if and only if $z_i = z_j$. For each partition λ of $[n]$, define the topological subspace

$$\mathbb{C}^\lambda = \{\mathbf{z} \in \mathbb{C}^n \mid \textsc{Part}(\mathbf{z}) = \lambda\}$$

of $\mathbb{C}^n$. First, observe that $\mathbb{C}^\lambda$ and $\mathbb{C}^\mu$ are disjoint if and only if λ and μ are distinct partitions, and that each point in $\mathbb{C}^n$ lies in a unique $\mathbb{C}^\lambda$. In other words, these sets form a partition of $\mathbb{C}^n$. Next, for each partition λ of $[n]$, the topological closure $\overline{\mathbb{C}^\lambda}$ is the linear subspace of $\mathbb{C}^n$ consisting of all points where we only require that $z_i = z_j$ if i and j belong to the same block. Thus, we see that $\overline{\mathbb{C}^\lambda}$ is the union of the disjoint subspaces $\mathbb{C}^\mu$, where μ is a partition of $[n]$ and λ is a refinement of μ. Finally, if $\mathbf{z} \in \mathbb{C}^\lambda$, then $\theta(\mathbf{z}) \in \overline{\mathbb{C}^\lambda}$.

Both $\mathbb{C}^\lambda$ and $\overline{\mathbb{C}^\lambda}$ are familiar topological spaces. Fix a partition $\lambda = \{S_1, \dots, S_\ell\}$ of $[n]$ and define a map $\overline{\varphi} : \overline{\mathbb{C}^\lambda} \to \mathbb{C}^\ell$ by sending each $\mathbf{z} \in \overline{\mathbb{C}^\lambda}$ to $\overline{\varphi}(\mathbf{z}) = (y_1, \dots, y_\ell)$, where each y_i is the common value shared by entries in $\mathbf{z}$ with indices from S_i. For example, if $\lambda = \{\{1,3\},\{2,5\},\{4\}\}$ and $\mathbf{z} = (a,b,a,b,b) \in \mathbb{C}^\lambda$, then we have

$\overline{\varphi}(\mathbf{z}) = (a, b, b)$. It is then clear that $\overline{\varphi}$ is a homeomorphism and the restriction of $\overline{\varphi}$ to $\mathbb{C}^\lambda$ is a homeomorphism onto its image: the space of all points in $\mathbb{C}^\ell$ with distinct coordinates. This image is well-known as the complement of the *complex braid arrangement*, i.e. the complement of the hyperplanes in $\mathbb{C}^\ell$ defined by the equations $y_i = y_j$. If we denote the union of these hyperplanes by $\mathcal{A}_\ell$, then we have that $\mathbb{C}^\lambda$ and $\overline{\mathbb{C}^\lambda}$ are homeomorphic via $\overline{\varphi}$ to $\mathbb{C}^\ell - \mathcal{A}_\ell$ and $\mathbb{C}^\ell$, respectively. We are now ready to prove Theorem C.

Proof of Theorem C. Let $\lambda = \{S_1, \ldots, S_\ell\}$ be a partition of $[n]$ with $a_i = |S_i|$ for each i and define the map $\overline{\varphi}$ as above. Then the restricted map $\varphi \colon \mathbb{C}^\lambda \to \mathbb{C}^\ell - \mathcal{A}_\ell$ is a homeomorphism, so we have that θ_λ is a local homeomorphism at $\mathbf{z} \in \mathbb{C}^\lambda$ if and only if the map $\varphi\theta_\lambda\varphi^{-1} \colon \mathbb{C}^\ell - \mathcal{A}_\ell \to \mathbb{C}^\ell$ is a local homeomorphism at $\varphi(\mathbf{z})$. If we define $\mathbf{a} = (a_1, \ldots, a_\ell)$, we then have that $\varphi\theta_\lambda\varphi^{-1}$ sends each $\mathbf{y} = (y_1, \ldots, y_\ell) \in \mathbb{C}^\ell - \mathcal{A}_\ell$ to $(p_{\mathbf{a}}(y_1), \ldots, p_{\mathbf{a}}(y_\ell))$. In other words, $\varphi\theta_\lambda\varphi^{-1} = \theta_{\mathbf{a}}$ and by Theorem B, the associated Jacobian matrix $\mathbf{J}_{\mathbf{a}}$ is invertible if and only if $y_1, \ldots, y_\ell$ are distinct and nonzero. Together with the Inverse Function Theorem, this implies that $\theta_{\mathbf{a}}$ is a local homeomorphism at $\mathbf{y} \in \mathbb{C}^\ell - \mathcal{A}_\ell$ if the entries of $\mathbf{y}$ are distinct and nonzero. By the definition of φ, we conclude that θ_λ is a local homeomorphism at $\mathbf{z} \in \mathbb{C}^\lambda$ if the entries of $\mathbf{z}$ are nonzero. □

Remark 6.3 (Lifting critical value motions)**.** The explicit local homeomorphism property described in Theorem C has an interesting consequence. One can show that, given a specific complex polynomial with distinct roots and a motion of its critical values, there is a unique lift of this motion to a subspace of polynomials with critical points that continue to be partitioned in the same fixed manner. This application is among the primary motivations for the current work, and an explicit statement will be stated and proved in a future article.

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Email address: `michael.dougherty@colby.edu`

Department of Mathematics and Statistics, Colby College, Waterville, ME 04901

Email address: `jon.mccammond@math.ucsb.edu`

Department of Mathematics, UC Santa Barbara, Santa Barbara, CA 93106